The Theory of Divisors in Krull Domains

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Contents

Chapter 1

Introduction

This chapter serves as an introduction to the paper. Its primary purpose is to motivate the discussion to follow, discuss the background that will be assumed, and provide an outline for what is to be discussed in the remaining chapters and sections. We begin with some reasons as to why the subject matter of divisors in a Krull domain is so appealing.

1.1 Motivation

One of the most basic theorems in number theory says that each integer can be uniquely expressed as a product of primes. This brings great joy to our lives. But can the idea be extended further? Certainly, the concept has a natural extension in the family of unique factorization domains, rings in which every non-zero element is the unique product of irreducible elements, up to unit multiples. And this also brings great joy to our lives. But what if a ring does not have unique factorization, how close to it can we get then? And so along comes the Dedekind domain, a ring in which every ideal factors uniquely into a product of prime ideals, preserving the idea of unique factorization at the mere cost of allowing factors which are not numbers, but rather ideals. And this, again, brings great joy to our lives. But can we produce a satisfactory theory of factorization on a wider class of rings? And how far are we from unique factorization of elements in these rings that allow unique factorization of ideals?

As the theory becomes more complex, more and more questions arise. The basic concepts developed for algebraic number theory also worked well in developing a theory of factorization for Dedekind domains, but the question of unique factorization applies to any domain. In fact, how can we even describe the degree of uniqueness of factorization in a general domain? We have seen that the size of the ideal class group can reasonably be used as such a meaure in Dedekind domains, but can this be generalized to a larger family of rings? And how complex does the factorization get, based on this notion of an ideal class group?

By a twist of fate, both of these questions are answered by the single concept of a Krull domain, rings formed as special intersections of discrete valuation rings. A superfamily of the family of Dedekind domains, Krull domains permit a theory of divisors that is very similar to the theory of fractional ideals developed for Dedekind domains. Through them, we define a divisor class group that parallels the definition of an ideal class group, and still has the property of measuring uniqueness of factorization. In particular, a Krull domain is a unique factorization domain if and only if its divisor class group is trivial. Deeper reflection upon their structure reveals a very close tie to Dedekind domains: namely, that Dedekind domains are exactly the dimension 1 Krull domains. Further study of the divisor class group then reveals an amazing result — that every abelian group occurs as the divisor class group of a Krull domain, and moreover, as the ideal class group of a Dedekind domain. Thus the study of Krull domains allows us to see that factorization of elements in the relatively well-behaved family of Dedekind domains is in fact arbitrarily complex. The fact that the family of Krull domains is closed under various natural algebraic transformations such as localizations and polynomial extensions only adds to the interest, confirming that they are a natural family of rings to study.

And thus, the theory of divisors in Krull domains arises out of curiosity about Dedekind domains, with strong ties to factorization and valuations of a field. Krull rings are primarily interesting because of their robust theory of factorization and their relationship to Dedekind domains, both of which rest strongly on the theory behind the group formed by their divisors. These divisors and their divisor class group have a rich theory of their own which we develop to the fullest. Upon completion, we will be able to see how the divisor class group can be used as an adequate measure of unique factorization, and demonstrate that any abelian group can be realized as the divisor class group of an appropriate Krull domain or Dedekind domain. Thus we will have formed a strong theory of factorization for a large class of rings, where factorization can become arbitrarily complex.

And this will also bring great joy to our lives.

1.2 Preliminaries

The purpose of this section is to set the context in which this paper is to be read. While the paper is self-contained, it makes various simple assumptions from the more general fields of mathematics which it is built upon; it is our purpose here to point these assumptions out. Here we detail the level of knowledge assumed, and in a few places point out definitions or theorems that will be used in the sequel.

The mathematical background used by this paper is purely algebraic. One can trace its origins through an increasingly narrower series of topics, from basic algebra, to commutative ring theory, to local field theory, to valuation theory, to Krull theory, to the theory of divisors in Krull domains. As such, it is natural for us to present the facts used and assumptions made at each of these levels.

Basic Algebra: We make free use of ideas from basic algebra, such as principal ideal domains, unique factorization domains, semigroups, congruence relations, free groups and bases, prime and maximal ideals, irreducibles, polynomial extensions, radicals, and direct sums. One basic lemma which is used quite often is the fact that any ideal contained in a finite union of prime ideals is contained within a single one of the prime ideals, i.e.

$$I \subseteq \bigcup_{i=1}^{m} P_i \Rightarrow I \subseteq P_i \text{ for some } i, 1 \leq i \leq m,$$

where I is an ideal and $P_1, \ldots, P_m$ are prime ideals. We use the notation $A \setminus B$ to denote the set difference $\{x \in A : x \notin B\}$, and R_0 to denote the fractional field of R.

Commutative Ring Theory: We assume familiarity with Noetherian rings and modules, integral and almost integral elements, normal and completely normal rings, integral extensions, algebras, exact sequences, tensor products, and flat modules. One theorem used without proof is the "Going Down" Theorem on extending chains of prime ideals from a normal domain R to an integral extension ring R'. We also use several basic facts concerning Noetherian rings and modules: Noetherian rings and modules are finitely generated, the direct sum of finitely many Noetherian modules is again Noetherian, and an R-module M with an R-submodule N is Noetherian if and only if N and M/N are both Noetherian. We refer the reader to Matsamura [5] for any definitions or proofs of theorems.

Local Field Theory: We assume familiarity with the prime spectrum $\mathrm{Spec}(R)$, the maximal spectrum m-$\mathrm{Spec}(R)$, heights of prime ideals and the extension to heights of arbitrary ideals, the set of height 1 prime ideals $\mathrm{Spec}^1(R)$, primary ideals, P-primary ideals, Krull dimension, localization at a prime ideal $P \in \mathrm{Spec}(R)$, and localization at a general multiplicative subset $S \subseteq R$.

The following are some theorems from local field theory which we use in the paper: The localization R_P is a local ring with unique maximal ideal PR_P; a ring is the intersection of its localizations at its maximal ideals; the spectrum of a localization is given by $\mathrm{Spec}(R_S) = \mathrm{Spec}(R) \setminus \{P \in \mathrm{Spec}(R) : P \cap S \neq \emptyset\}$, i.e. the prime ideals in a localization of R are those prime ideals of R which do not intersect the multiplicative subset, and in this case the relation of inclusion is preserved. In particular, the same theorem then applies to the set of height 1 prime ideals.

Another very important subset of local field theory from which we borrow freely is the theory of Dedekind domains; indeed, the main result of the paper is dedicated to them. We assume that the reader is familiar with the definition of Dedekind domains, fractional ideals, principal fractional ideals, the ideal class group, and the set $I : J = \{x \in R_0 : xJ \subseteq I\}$, where I and J are fractional ideals of the domain R. We use the fact that every fractional ideal factors uniquely into a product of prime ideals, as well as the fact that the localization of a Dedekind domain at one of its prime ideals is a principal ideal domain.

Valuation Theory: We assume the definitions of valuations, value groups, valuation rings, and discrete valuation rings, as well as the basic conversions between valuations and their corresponding valuation rings; we denote the valuation ring of a valuation v by R_v. We also use the fact that every valuation ring is local (we denote its unique maximal ideal by $\mathfrak{m}(R_v)$), and that a discrete valuation ring is equivalent to a Noetherian valuation ring or a local principal ideal domain.

We refer the reader to Serre [7] for more information on either local field theory (in particular the section on Dedekind domains) or valuation theory.

Krull Theory: At this point the theory is relatively specific, so we have included a chapter dedicated to defining Krull domains and presenting some of their properties which are to be used in the remainder of the paper. On the other hand, many of these facts are simple enough to prove or far enough removed from the theory of divisors that their proofs have been omitted. We refer the reader to Balcerzyk [1] or Krull [4] for the basic proofs involved.

The Theory of Divisors in Krull Domains: This is the focus of the paper. Barring the details mentioned in the preceding paragraphs, a complete presentation of the topic is given in the sequel. The proofs of the theorems involved are taken from Balcerzyk [1] and Fossum [3], with original work taken from Claborn [2] and Nagata [6]. A comprehensive list of the specific concepts covered are detailed in the next section.

1.3 Goals

Now that we have prepared the reader for the content and background of the paper, we continue by listing the goals we hope to achieve at each level of the presentation.

The goal of this paper is to present a thorough discussion of the theory of divisors in Krull domains. This ranges from the definitions of divisors and the divisor class group, to their basic properties and profound theorems, to a powerful application which proves that every abelian group occurs as the ideal class group of a Dedekind domain. Here are the goals of each chapter and section in more detail:

Chapter 1: Introduction. The goal is to prepare the reader for the material to be covered. Section 1.1 motivates the paper. Section 1.2 details the assumptions made throughout the paper. Section 1.3 outlines the goals of the paper and each of its chapters and sections. Section 1.4 gives acknowledgments.

Chapter 2: Krull Rings. The goal is to review the definition and basic properties of a Krull domain. Section 2.1 presents the several equivalent definitions of a Krull domain. Section 2.2 describes some of the basic properties satisfied by a Krull domain, including closures under algebraic transformations, descriptions of the set of height 1 prime ideals, and complete normality of a Krull domain. Section 2.3 gives examples of families of rings which are Krull domains, and proves that a ring is a Dedekind domain if and only if it is a Krull domain of dimension 1.

Chapter 3: Divisors. The goal is to define divisors and cover two important results related to divisors in a Krull domain: the fact that the set of divisors of a Krull domain forms a free abelian group based on its height 1 prime ideals, and the fact that every divisor is the intersection of two principal divisors. Section 3.1 provides the definition of divisors in a general domain, as well as some of the basic properties they satisfy. Section 3.2 specifies important properties of divisors of a Krull domain, most significantly the theorem that the smallest divisor containing a fractional ideal I is the intersection of its extensions IR_P to the localizations of R at its height 1 prime ideals P. Section 3.3 proves that the set of divisors of a Krull domain R is the free $\mathbb{Z}$-module $\mathbb{Z}\{\mathrm{Spec}^1(R)\}$. Section 3.4 proves that every divisor is the intersection of two principal divisors through another important result, the Approximation Theorem for Krull domains.

Chapter 4: The Divisor Class Group. The goal is to define the divisor class group, determine its importance, and relate the class group of a Krull domain to those of other Krull domains which are algebraically related; the latter is very useful for the next chapter, which involves constructing rings with particular class groups. Section 4.1 provides the definition and some basic properties of the semigroup of divisor classes in general domains; through this, it defines the divisor class group for Krull domains. Section 4.2 proves that a ring is a unique factorization domain if and only if it is a Krull domain with divisor class group 0, providing the motivation for using the size of a Krull domain's divisor class group as a measure of the uniqueness of factorization within it. Section 4.3 investigates when an extension of Krull domains induces a homomorphism of their class groups, and proves the result for a restricted but useful family of extensions. Section 4.4 proves Nagata's Theorem, which describes the kernel of an induced homomorphism for certain classes of extensions. Section 4.5 provides examples of algebraic constructions which yield Krull domains with isomorphic divisor class groups.

Chapter 5: Constructions of Rings with Specified Class Groups. The goal is to prove Claborn's Theorem, a powerful application of Krull theory which states that every abelian group occurs as the ideal class group of a Dedekind domain. Section 5.1 gives a construction for a Krull domain whose divisor class group is a specified free abelian group. Section 5.2 gives a construction for a Krull domain whose divisor class group is the quotient group G/H, where G is the divisor class group of a given Krull domain and H is any subgroup thereof. Section 5.3 gives a construction for a Dedekind domain whose ideal class group is isomorphic to the divisor class group of a specified Krull domain. Section 5.4 proves the main theorem, that every abelian group occurs as the divisor class group of a Krull domain and the ideal class group of a Dedekind domain.

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Chapter 2

Krull Rings

In this chapter we review the definition and basic properties of Krull domains. We begin by giving several equivalent definitions of a Krull domain, add some related definitions, and then proceed to describe some of the basic properties of Krull theory. Along the way, we discuss properties of closure under simple algebraic transformations, describe $\mathrm{Spec}^1(R)$ for certain Krull domains R, give examples of Krull domains, and prove that a ring is a Dedekind domain if and only if it is a Krull domain of dimension 1. The proofs of many of these facts are omitted since they are relatively simple and not the main focus of this paper.

2.1 Definitions

We begin with the definition of a Krull domain.

Definition 2.1.1. A **Krull domain** is a domain R which satisfies the following conditions:

1. $R = \bigcap\limits_{\lambda \in \Lambda} V_\lambda$, where $\{V_\lambda\}_{\lambda \in \Lambda}$ is a family of discrete valuation rings over a field K, and

2. Every non-zero element $r \in R$ is invertible in V_λ for almost all $\lambda \in \Lambda$.

Condition 1 is called the **intersection property**, and condition 2 is called the **finite character property**. $\{V_\lambda\}_{\lambda \in \Lambda}$ is called the **defining family** of R.

The definition of a Krull domain first appeared in Krull [4] in 1931. In his original article, Krull called a ring satisfying the properties listed above a **finite discrete principal order**.

The following proposition derives from an equivalent definition for Krull domains.

Proposition 2.1.1. *A domain R is a Krull domain if and only if it satisfies the following conditions:*

1. *R_P is a discrete valuation ring for every prime ideal $P \in \mathrm{Spec}^1(R)$,*

2. *$R = \bigcap\limits_{P \in \mathrm{Spec}^1(R)} R_P$, and*

3. *Every non-zero element $r \in R$ is invertible in R_P for almost all $P \in \mathrm{Spec}^1(R)$.*

Other definitions are sometimes given based on some simple variants of the conditions given in Proposition 2.1.1; we describe a few of these here and use them all interchangeably.

From alternate definitions of discrete valuation rings, condition 1 of Proposition 2.1.1 is equivalent to each of the following conditions:

1. R_P is a Noetherian valuation ring for every prime ideal $P \in \mathrm{Spec}^1(R)$,
2. R_P is a local principal ideal domain for every prime ideal $P \in \mathrm{Spec}^1(R)$.

From properties of localizations, condition 3 of Proposition 2.1.1 is equivalent to each of the following conditions:

1. Every non-zero element $r \in R$ belongs to only finitely many $P \in \mathrm{Spec}^1(R)$,
2. Every non-zero element $x \in R_0$ is an invertible element of R_P for almost all $P \in \mathrm{Spec}^1(R)$.

From properties of primary ideals, conditions 2 and 3 of Proposition 2.1.1 are equivalent to the following condition:

1. For every non-zero non-unit element $r \in R$ there exist height 1 primary ideals $Q_1, \ldots, Q_m$ such that $Rr = \bigcap\limits_{i=1}^{m} Q_i$.

This formulation is useful because it states the definition in terms of principal ideals of R, as opposed to the set of height 1 prime ideals used in the previous definitions.

This concludes our discussion of the various definitions of a Krull domain. Next, we investigate some of their basic properties.

2.2 Basic Properties

We begin this section with a few definitions related to Krull domains.

Definition 2.2.1. An **irredundant representation** of a Krull domain R is a representation $R = \bigcap\limits_{\lambda \in \Lambda} V_\lambda$, where $\bigcap\limits_{\lambda \in \Lambda'} V_\lambda \neq \bigcap\limits_{\lambda \in \Lambda} V_\lambda$ for every proper subset $\Lambda' \subsetneq \Lambda$.

In simple terms, an irredundant representation is a representation in which every term of the intersection contributes to the result, i.e. every term excludes some element which every other term contains. It is clear that every Krull domain has an irredundant representation, yielding a minimal defining family.

Definition 2.2.2. A **subintersection** of a Krull domain R is a ring $R(\Lambda) = \bigcap\limits_{P \in \Lambda} R_P$, where $\Lambda \subseteq \mathrm{Spec}^1(R)$ is a set of height one prime ideals of R.

Thus, a subintersection of a Krull domain R is a ring whose defining family is a subset of the defining family of R.

It is clear from the definition of a Krull domain that arbitrary intersections of Krull domains are again Krull domains. As it turns out, there are various ways of constructing Krull domains from other Krull domains, many of which turn out to be quite useful. Here we give a sampling of some of the more common algebraic transformations under which Krull domains are closed.

Proposition 2.2.1. *Let R be a Krull domain and S a multiplicative subset of R. Then the localization R_S is a Krull domain.*

Proposition 2.2.2. *Let R be a Krull domain and $\Lambda \subseteq \operatorname{Spec}^1(R)$ a set of height one prime ideals of R. Then the subintersection $R(\Lambda) = \bigcap_{P \in \Lambda} R_P$ is a Krull domain.*

Proposition 2.2.3. *Let R be a Krull domain and $\{X_\gamma\}_{\gamma\in\Gamma}$ an arbitrary set of indeterminates. Then the polynomial extension ring $R[\Gamma] = R[\{X_\gamma\}_{\gamma\in\Gamma}]$ is a Krull domain.*

There are other transformations which yield Krull domains as well, such as extension to a power series ring, intersection with the subfield of a superfield, or normalization in a finite extension field of the field of fractions. We omit these as propositions because they are not necessary to the theory we develop in the sequel.

Our next focus is a description of height 1 prime ideals in certain Krull domains. Our first result gives a description of these ideals in a polynomial extension ring like the one described in the last proposition.

Proposition 2.2.4. *Let R be a Krull domain, $R[\Gamma] = R[\{X_\gamma\}_{\gamma\in\Gamma}]$ a ring of polynomials, and $Q \in \operatorname{Spec}^1(R[\Gamma])$. Then Q is of the form*

1. *$Q = R_0[\Gamma]q \cap R[\Gamma]$ for some irreducible element $q \in R_0[\Gamma]$ (in the case that $Q \cap R = 0$), or*

2. *$Q = PR[\Gamma]$ for some $P \in \operatorname{Spec}^1(R)$ (in the case that $Q \cap R \neq 0$)*

Conversely, sets of the form 1 or 2 are members of $\operatorname{Spec}^1(R[\Gamma])$.

We note that if Q is of form 1 then $R_0[\Gamma]q = QR_0[\Gamma] = QR_0 \in \operatorname{Spec}^1 R_0[\Gamma]$, and if Q is of form 2 then $P = Q \cap R$.

In the case that we have an irredundant representation for a Krull domain, we have a much stronger description. The following proposition gives a bijection between the set of height 1 prime ideals of a Krull domain and the defining family of its irredundant representation.

Proposition 2.2.5. *Let R be a Krull domain and $R = \bigcap_{\lambda\in\Lambda} V_\lambda$ an irredundant representation of R. Then there is a bijection between Λ and $\operatorname{Spec}^1(R)$ given by the map $\phi(\lambda) = \mathfrak{m}(V_\lambda) \cap R$.*

Our final proposition is self-explanatory.

Proposition 2.2.6. *Let R be a Krull domain. Then R is completely normal.*

2.3 Examples

In this section, we finally give some examples of Krull domains. Because it is rather unenlightening to construct complicated examples of Krull domains which have few interesting properties, we instead give examples of other well-behaved families of rings that lie within the family of Krull domains.

We begin with the family of unique factorization domains.

Theorem 2.3.1. *Let R be a unique factorization domain. Then R is a Krull domain.*

PROOF. We note that R is Noetherian since it is a unique factorization domain. For each $P \in \operatorname{Spec}^1(R)$, take a non-zero element of P and factor it into irreducibles; one of the irreducible factors p must also be an element of P by primality. Then the ideal $Rp \subseteq P$ is prime, so it must be equal to P since the latter is height 1. Then the ring R_P consists of all elements $\frac{r}{s} \in R_0$ such that $p \nmid s$; this is clearly a valuation ring, and is Noetherian because it is the localization of a Noetherian ring. Thus, each R_P is a discrete valuation ring. It is also clear from this description that

$$R = \bigcap_{P \in \operatorname{Spec}^1(R)} R_P$$

since every fraction in R_0 with a non-unit denominator is excluded by the localization at the prime ideal generated by any of its irreducible divisors. Finally, we note that every non-zero element is invertible in almost all of the R_P: namely, all those which are not generated by any of the finite number of irreducibles that divide it. It follows that R is a Krull domain. $\square$

Thus, all unique factorization domains are Krull domains. In particular, principal ideal domains and fields are Krull domains, as well as arbitrary polynomial extensions of either. Later, once we have further developed the theory of divisors, we shall determine a condition on a Krull domain which is necessary and sufficient for it to be a unique factorization domain.

Next, we consider the family of Dedekind domains.

Theorem 2.3.2. *Let R be a Dedekind domain. Then R is a Krull domain.*

PROOF. First, we recall that for any prime ideal P of the Dedekind domain R, the localization R_P is a principal ideal domain; in particular, this holds for the height 1 prime ideals. Since these localizations are also local rings, they are local principal ideal domains and thus discrete valuation rings as desired.

Second, we recall that a ring is the intersection of its localizations at its maximal ideals, i.e.

$$R = \bigcap_{\mathfrak{m} \in \mathfrak{m}\text{-}\operatorname{Spec}(R)} R_{\mathfrak{m}}.$$

Since R is a Dedekind domain, where all non-zero prime ideals are maximal, we have that m-Spec$(R) = \text{Spec}(R) = \text{Spec}^1(R)$, so

$$R = \bigcap_{P \in \text{Spec}^1(R)} R_P$$

as desired.

Finally, we need to show that every non-zero element $r \in R$ is contained in only finitely many height 1 prime ideals. To see this, consider the unique factorization of the ideal Rr into products of prime ideals. Each of the factors is P-primary for some $P \in \text{Spec}^1(R)$, and clearly r is contained only in those $P \in \text{Spec}^1(R)$ for which there is a P-primary factor of Rr. This is only finitely many $P \in \text{Spec}^1(R)$ since the factorization is finite, so the result is proven. □

Thus, every Dedekind domain is a Krull domain. As with unique factorization domains, we can be even more specific about conditions sufficient for the converse to hold. Our next theorem accomplishes this by revealing the exact relationship between Krull domains and Dedekind domains.

Theorem 2.3.3. *Let R be a ring. Then the following conditions are equivalent:*

1. *R is a Dedekind domain.*
2. *R is a Krull domain of dimension* 1.

Proof. We show that each condition implies the other.

$1 \Rightarrow 2$: Let R be a Dedekind domain. We have already shown in Theorem 2.3.2 that R is a Krull domain. The fact that it is dimension 1 follows directly from the definition of a Dedekind domain.

$2 \Rightarrow 1$: Let R be a Krull domain of dimension 1. To show that R is a Dedekind domain, we need to show that it is normal, dimension 1, and Noetherian. The fact that R is normal follows from Proposition 2.2.6; the fact that it is dimension 1 is part of our assumption. Thus all we have left to prove is that it is Noetherian.

The ideal R is clearly finitely generated; let I be a non-zero proper ideal. Given a non-zero element $r \in I$, r is non-invertible, so by the definition of a Krull domain there exist height 1 primary ideals $Q_1, \ldots, Q_m$ such that $Rr = \bigcap_{i=1}^{m} Q_i$. For each Q_i, $Rr \subseteq Q_i$, so there is a natural projection map $\phi_i : R/Rr \to R/Q_i$ with $\text{Ker}(\phi_i) = Q_i/Rr$; we put these together to form the homomorphism

$$\phi : R/Rr \to \bigoplus_{i=1}^{m} R/Q_i,$$

where $\text{Ker}(\phi) = \bigcap_{i=1}^{m} Q_i/Rr = \left(\bigcap_{i=1}^{m} Q_i\right)/Rr = Rr/Rr = 0$. Thus ϕ is an injection, so $I/Rr \subseteq R/Rr$ can be viewed as an R-submodule of $\bigoplus_{i=1}^{m} R/Q_i$. Now it suffices to show that for every height 1 primary ideal Q, R/Q is Noetherian; from this it would follow that each R/Q_i is Noetherian, and thus their direct sum $\bigoplus_{i=1}^{m} R/Q_i$ would be Noetherian, and thus its submodule I/Rr would be Noetherian, and thus I would be Noetherian (since Rr is principal and hence Noetherian), and thus I would be finitely generated, whence R would be Noetherian as desired.

Let Q be a height 1 primary ideal of R. Then $P = \mathrm{rad}(Q)$ is a prime ideal, and hence a maximal ideal, of R, so P/Q is a maximal ideal of R/Q. In fact, we note that any non-zero element $r + Q \in R/Q \setminus P/Q$ is a unit of R/Q: $r + Q \notin P/Q \Rightarrow r \notin P$, so r and P generate R by maximality; in particular, we can write $1 = rs + p$ for some $s \in R, p \in P$; but then $p^n \in Q$ for some positive integer n, so $1 - rr' = 1 - r(\frac{1-(1-rs)^n}{r}) = (1 - rs)^n = p^n \in Q$, where $r' = \frac{1-(1-rs)^n}{r} \in R$; it follows that $(r + Q)(r' + Q) = rr' + Q = 1 + Q$, so that $r' + Q$ is the inverse of $r + Q$ in R/Q, whence r is a unit as desired. Thus, P/Q is in fact unique as a maximal ideal, whence R/Q is a local ring.

Now, the localization of a local ring at its maximal ideal is the whole ring, since everything outside the maximal ideal must be a unit. Thus, we have $R/Q \cong (R/Q)_{P/Q}$. The latter is isomorphic to R_P/QR_P by the natural homomorphism taking $\frac{r+Q}{s+Q} \mapsto \frac{r}{s} + QR_P$, (note that for this map to be well-defined, it is crucial that $Q \subseteq P$). Thus we have $R/Q \cong R_P/QR_P$. But by the definition of a Krull domain, each R_P is Noetherian, whence the quotient ring R_P/QR_P is Noetherian as well. Thus R/Q is Noetherian as desired, and the result follows. $\square$

As a final note, we add that it is also true that any normal Noetherian domain is a Krull domain, encompassing both of the cases above. The proof is not difficult, and is relatively similar to those given above. We omit it because of this similarity, and because it is of little relevance to the theory of divisors we develop in the remainder of the paper.

Chapter 3

Divisors

The purpose of this chapter is to define divisors — the fundamental unit of our study — and develop the foundation of the theory of divisors in a Krull domain: concepts dealing with the divisors themselves. After giving definitions and developing the basic properties, we prove two important results. In section 3.3, we present a theorem that the set of divisors of a Krull domain R is isomorphic to a free abelian group based on its height 1 prime ideals. In section 3.4, we present a theorem that every divisor is the intersection of two principal divisors.

3.1 Definition and Basic Properties

We begin with some definitions. We omit the proofs of simple propositions which may be easily reconstructed.

Definition 3.1.1. A **divisor** of a domain R is a fractional ideal $I \subseteq R$ which is the intersection of a set of principal fractional ideals.

Divisors are also known as **divisorial ideals**. We denote the set of non-zero fractional ideals of R by $\mathrm{Id}(R)$, the set of divisors of R by $\mathrm{Div}(R)$, and the set of principal divisors (also the set of principal fractional ideals) of R by $\mathrm{Prin}(R)$.

It follows directly from the definition that a divisor I is the intersection of all the principal fractional ideals which contain it. We generalize this notion to form a function that takes fractional ideals to divisors.

Definition 3.1.2. Let R be a domain, $I \in \mathrm{Id}(R)$, and $D_I = \{x \in R_0 : I \subseteq Rx\}$. Then we define a map $\mathrm{div} : \mathrm{Id}(R) \to \mathrm{Div}(R)$ by

$$\mathrm{div}(I) = \bigcap_{x \in D_I} Rx.$$

From the above discussion, we have that $\mathrm{div}(I) = I$ for all $I \in \mathrm{Div}(R)$. In fact, it is also quite clear that this is a sufficient condition to ensure that $I \in \mathrm{Div}(R)$, since $I = \mathrm{div}(I) = \bigcap_{x \in D_I} Rx$ is an intersection of principal fractional ideals. We sum up this fact and some other simple properties in the following proposition.

Proposition 3.1.1. *Let R be a domain and $I, J, I', J' \in \mathrm{Id}(R)$. Then*

1. $\mathrm{div}(I) = I \;\Leftrightarrow\; I \in \mathrm{Div}(R)$,
2. $I \in \mathrm{Div}(R) \Rightarrow I : J \in \mathrm{Div}(R)$,
3. $\mathrm{div}(I) = R : (R : I)$,
4. $\mathrm{div}(I) = \mathrm{div}(J) \;\Leftrightarrow\; R : I = R : J$, *and*
5. $\mathrm{div}(I) = \mathrm{div}(I'), \mathrm{div}(J) = \mathrm{div}(J') \Rightarrow \mathrm{div}(IJ) = \mathrm{div}(I'J')$.

Now, we have a set of divisors $\mathrm{Div}(R)$ for the domain R. We would also like to have a binary operation on the set of divisors that would give it a nice structure. Unfortunately, products of divisors are not necessarily divisors. Thus, to retain closure we define the operation

$$I + J = \mathrm{div}(IJ),$$

where $I, J \in \mathrm{Div}(R)$. It is clear that this operation is well-defined, commutative, associative, and has an identity element R; hence, it gives $\mathrm{Div}(R)$ the structure of a commutative semigroup with zero, referred to as the **semigroup of divisors**. We also note that the final item of Proposition 3.1.1 shows that this operation is compatible with the map $\mathrm{div} : \mathrm{Id}(R) \to \mathrm{Div}(R)$, i.e.

$$\mathrm{div}(I) + \mathrm{div}(J) = \mathrm{div}(IJ),$$

for all $I, J \in \mathrm{Id}(R)$.

The natural question to ask next is when this semigroup is actually a group. It turns out that the answer is relatively simple.

Proposition 3.1.2. *Let R be a domain. Then the semigroup of divisors $\mathrm{Div}(R)$ is a group if and only if R is completely normal.*

We observe in particular that $\mathrm{Div}(R)$ forms a group for all Krull domains R, since these are completely normal by Proposition 2.2.6. We refer to this as the **group of divisors**.

3.2 Divisors in Krull Rings

In this section we take the definition we have just given for the divisor of a general domain and tailor it to that of a Krull domain. There are various extra properties that we get in this special case, most notably the relationship between $\mathrm{div}(I)$ and the extensions of I in the height 1 prime localizations of R. We present this relationship as the culminating fact of the section, Theorem 3.2.5.

First, we must prove several lemmas that will be of use.

Lemma 3.2.1. *Let R be a Krull domain, $P_0, P_1, \dots, P_m \in \mathrm{Spec}^1(R)$ distinct prime ideals. Then there exists $\pi \in P_0 \diagdown \bigcup_{i=1}^{m} P_i$ such that $\pi R_{P_0} = P_0 R_{P_0}$, i.e. π is a generator of $P_0 R_{P_0}$ in R_{P_0}. Note that we then have $v_{P_0}(\pi) = 1$, and $v_{P_i}(\pi) = 0$ for each $i, 1 \le i \le m$.*

Proof. From the definition of a Krull domain, R_P is a principal ideal domain, so we can find $r \in R_P$ a generator of the ideal PR_P; through multiplication by a suitable element of R we can assume that $r \in R$. Re-indexing the P_i if necessary, suppose that $r \in P_1, \ldots, P_k$, and $r \notin P_{k+1}, \ldots, P_m$. We know that $P_i \setminus \bigcup_{j=1}^{k} P_j$ is non-empty for $i = 0, k+1, k+2, \ldots, m$ by distinctness (otherwise, $P_i \subseteq \bigcup_{j=1}^{k} P_j \Rightarrow P_i \subseteq P_j$ for some $j, 1 \le j \le k$, but this is impossible because the P_i are distinct height 1 prime ideals), so for each such i we choose $r_i \in P_i \setminus \bigcup_{j=1}^{k} P_j$. Setting

$$r' = r_0^2 \prod_{j=k+1}^{m} r_j,$$

we have $r' \in P_0^2$ and $r' \in P_i$ for each $i, k+1 \le i \le m$, but $r' \notin P_i$ for each $i, 1 \le i \le k$. Setting $\pi = r - r'$ yields $\pi \in P_0$ but $\pi \notin P_0^2$ (so that π generates PR_P), and $\pi \notin P_i$ for each $i, 1 \le i \le m$; the result follows.

The note follows from the facts that invertible elements have value 0 and that any generator of the unique maximal ideal of a valuation ring has minimal positive valuation value (1 in the case of a discrete valuation ring) since no distinct ideals lie between the whole ring (value 0) and the maximal ideal. $\square$

Lemma 3.2.2. *Let R be a Krull domain, $P \in \mathrm{Spec}^1(R)$, and $I \in \mathrm{Id}(R_P)$. Then $I = r_0^k R_P = (PR_P)^k$ for some $k \in \mathbb{Z}$, where r_0 is a generator of PR_P.*

Proof. Recall that R_P is a principal ideal domain; since $aI \subseteq R_P$ for an appropriate non-zero $a \in R_P$, $aI = bR_P$ for some non-zero $b \in R_P$; it follows that $I = \frac{a}{b} R_P$. Thus, it suffices to show that each non-zero element $r \in R$ is uniquely expressible as $r_0^k s$, where $s \notin P$ is invertible in R_P (this property then extends to R_0, in particular for $\frac{a}{b}$, and the result follows). To see this, note that r is expressible in this fashion if and only if it is an element of $r_0^k R_P = (r_0 R_P)^k = (PR_P)^k$ but not of $(PR_P)^{k+1}$. Since sets of the form $(PR_P)^k \setminus (PR_P)^{k+1}$ partition $R_P \setminus 0 \supseteq R \setminus 0$, $r \neq 0$ certainly falls uniquely into one of these sets as desired. $\square$

Note that this uniquely defines any discrete valuation v_P with valuation ring R_P to a multiple of the natural valuation, which sends x to the integer k corresponding to the ideal xR_P.

Lemma 3.2.3. *Let R be a Krull domain and $I \in \mathrm{Id}(R)$. Then $IR_P = R_P$ for almost all $P \in \mathrm{Spec}^1(R)$.*

Proof. Take non-zero $r \in R$ such that $rI \subseteq R$ and non-zero $a \in I$. Each of r and a are invertible in almost all R_P; thus they are both invertible in almost all R_P as well. For these

R_P, we have

$$\begin{aligned} R_P &\supseteq rIR_P \text{ (since } rI \subseteq R) \\ &= IR_P \text{ (by invertibility of } r) \\ &\supseteq raR_P \text{ (since } ra \in I) \\ &= R_P \text{ (by invertibility of } ra), \end{aligned}$$

whence $IR_P = R_P$ as desired. $\square$

Lemma 3.2.4. *Let R be a Krull domain, $P \in \operatorname{Spec}^1(R)$, and $I \in \operatorname{Id}(R)$. Then $(R : I)R_P = R_P : IR_P$.*

Proof. We show that each set is contained in the other.

$\subseteq$: $xI \subseteq R \Rightarrow xIR_P \subseteq RR_P = R_P$, so $R : I \subseteq R_P : IR_P$; since R_P is closed under multiplication from itself we can extend this to $(R : I)R_P \subseteq R_P : IR_P$. This holds for all $P \in \operatorname{Spec}^1(R)$.

$\supseteq$: Consider non-zero $x \in R_P : IR_P$; this satisfies $xIR_P \subseteq R_P$. By Lemma 3.2.3, $xIR_{P'} = R_{P'}$ for almost all $P' \in \operatorname{Spec}^1(R)$; in particular, these satisfy $xIR_{P'} \subseteq R_{P'}$. Let $P_1, \dots, P_m$ be the height 1 prime ideals for which $xIR_{P'} \nsubseteq R_{P'}$ (note that these are all distinct from P). Then by Lemmas 3.2.1 and 3.2.2, we have $xIR_{P_i} = r_i^i k_i R_{P_i}$, where we can choose generators $r_i \in P_i \setminus P$. Setting $r = \prod_{i=1}^{m} r_i^{k_i}$, we have $r \in R \setminus P$ and $rxIR_{P_i} \subseteq R_{P_i}$ for all $P_i, 1 \leq i \leq m$, and thus $rxIR_{P'} \subseteq R_{P'}$ for all $P' \in \operatorname{Spec}^1(R)$. In particular, the subideal $rxI \subseteq R_{P'}$ for all $P' \in \operatorname{Spec}^1(R)$, whence $rxI \subseteq \bigcap_{P' \in \operatorname{Spec}^1(R)} R_{P'} = R$. Thus, $rx \in R : I$. We also have that $r \in R \setminus P$, whence $\frac{1}{r} \in R_P$, so $x = (rx)\frac{1}{r} \in (R : I)R_P$. Therefore $R_P : IR_P \subseteq (R : I)R_P$ as desired. $\square$

Finally, we have built up enough machinery to prove our main result for this section.

Theorem 3.2.5. *Let R be a Krull domain and $I \in \operatorname{Id}(R)$. Then*

$$\operatorname{div}(I) = \bigcap_{P \in \operatorname{Spec}^1(R)} IR_P.$$

Proof. We show that each set is contained in the other.

$\subseteq$: Using the property that $\operatorname{div}(I) = R : (R : I)$ twice, applying Lemma 3.2.4 twice in between, and then observing that every ideal in R_P is a divisor because R_P is a principal ideal domain, we get the following chain of equality:

$$\begin{aligned} (\operatorname{div}(I))R_P &= (R : (R : I))R_P \\ &= R_P : ((R : I)R_P) \\ &= R_P : (R_P : IR_P) \\ &= \operatorname{div}(IR_P) \\ &= IR_P. \end{aligned}$$

It follows that $\operatorname{div}(I) \subseteq (\operatorname{div}(I))R_P = IR_P$ for all $P \in \operatorname{Spec}^1(R)$, so $\operatorname{div}(I) \subseteq \bigcap_{P \in \operatorname{Spec}^1(R)} IR_P$ as desired.

$\supseteq$: Recall that $\operatorname{div}(I) = \bigcap_{x \in R_0, I \subseteq Rx} Rx$. Then for each $x \in R_0, I \subseteq Rx$, we have $R_P x = (Rx)R_P \supseteq IR_P$, whence

$$Rx = \left(\bigcap_{P \in \operatorname{Spec}^1(R)} R_P \right) x = \bigcap_{P \in \operatorname{Spec}^1(R)} R_P x \supseteq \bigcap_{P \in \operatorname{Spec}^1(R)} IR_P.$$

It follows that $\operatorname{div}(I) = \bigcap_{x \in R_0, I \subseteq Rx} Rx \supseteq \bigcap_{P \in \operatorname{Spec}^1(R)} IR_P$ as desired. □

Thus, we have proven the main theorem for the section. We note one immediate corollary that is of some interest.

Corollary 3.2.6. *Let R be a Krull domain and $I \in \operatorname{Spec}^1(R)$. Then $I \in \operatorname{Div}(R)$.*

3.3 The Divisor Group as the Free $\mathbb{Z}$-module $\mathbb{Z}\{\operatorname{Spec}^1(R)\}$

In this section we prove our first important result, the isomorphism between $\operatorname{Div}(R)$ and $\mathbb{Z}\{\operatorname{Spec}^1(R)\}$.

We begin by viewing elements in $\mathbb{Z}\{\operatorname{Spec}^1(R)\}$ as functions $\mu : \operatorname{Spec}^1(R) \to \mathbb{Z}$ which are equal to zero almost everywhere. We establish a strong lemma whose statement gives a foretaste of the map to be used for our isomorphism shortly thereafter.

Lemma 3.3.1. *Let R be a Krull domain and $\mu \in \mathbb{Z}\{\operatorname{Spec}^1(R)\}$. Let*

$$I = \{x \in R_0 : v_P(x) \geq \mu(P) \text{ for all } P \in \operatorname{Spec}^1(R)\} \cup \{0\}.$$

Then $IR_P = (PR_P)^{\mu(P)}$ for all $P \in \operatorname{Spec}^1(R)$.

Proof. We first show the result for non-negative functions μ. Fixing $P_0 \in \operatorname{Spec}^1(R)$, suppose $\mu(P) = 0$ for $P \neq P_1, \ldots, P_m$. Let r_0 be a generator of $P_0 R_{P_0}$ and choose $r_i \in P_i \setminus P_0 \subseteq R$ for each $i, 1 \leq i \leq m$. Setting $r = \prod_{i=0}^{m} r_i^{\mu(P_i)}$ yields $v_{P_0}(r) = \mu(P_0)$ (since every factor other than r_0 is invertible in R_{P_0}) and $v_{P_i}(r) = \sum_{i=0}^{m} \mu(P_i) v_{P_i}(r_i) \geq \mu(P_i)$ for every other $i, 1 \leq i \leq m$; thus $r \in I$. Then $rR_{P_0} = r_0^{\mu(P_0)} R_{P_0}$ (since the other factors are invertible) $= (r_0 R_{P_0})^{\mu(P_0)} = (P_0 R_{P_0})^{\mu(P_0)}$, so

$$\begin{aligned}
(P_0 R_{P_0})^{\mu(P_0)} &= rR_{P_0} \\
&\subseteq IR_{P_0} \\
&= \left(\bigcap_{P \in \operatorname{Spec}^1(R)} (PR_P)^{\mu(P)} \right) R_{P_0} \text{ (by the definition of } I \text{ and Lemma 3.2.2)} \\
&\subseteq (P_0 R_{P_0})^{\mu(P_0)} R_{P_0} \\
&= (P_0 R_{P_0})^{\mu(P_0)}.
\end{aligned}$$

Thus, $IR_{P_0} = (P_0 R_{P_0})^{\mu(P_0)}$ as desired.

Next, we consider functions with negative values. Suppose that $\mu(P) < 0$ for $P = P_1, \dots, P_m$, $\mu(P) \geq 0$ for $P \neq P_1, \dots, P_m$. Let $r_i \in R$ generate $P_i R_{P_i}$ for each $i, 1 \leq i \leq m$. Setting $r = \prod_{i=1}^{m} r_i^{-\mu(P_i)} \in R$ yields

$$\begin{aligned} rI &= \{rx \in R_0 : v_P(x) \geq \mu(P) \text{ for all } P \in \mathrm{Spec}^1(R)\} \cup r\{0\} \\ &= \{x \in R_0 : v_P\left(\frac{x}{r}\right) \geq \mu(P) \text{ for all } P \in \mathrm{Spec}^1(R)\} \cup \{0\} \\ &= \{x \in R_0 : v_P(x) \geq \mu(P) + v_P(r) \text{ for all } P \in \mathrm{Spec}^1(R)\} \cup \{0\}. \end{aligned}$$

Now, $\mu(P) + v_P(r)$ is non-negative for all $P \in \mathrm{Spec}^1(R)$ and the value is 0 for almost all P (since $\mu(P)$ and $v_P(r)$ are independently non-zero for only finitely many P), so we may apply the result of the preceding part of the proof to obtain $(rI)R_P = (PR_P)^{\mu(P)+v_P(r)}$ for all $P \in \mathrm{Spec}^1(R)$. Then $(PR_P)^{v_P(r)}(IR_P) = (rR_P)(IR_P)$ (this follows directly from the definition of v_P) $= (rI)R_P = (PR_P)^{\mu(P)+v_P(r)}$, whence $IR_P = (PR_P)^{\mu(P)}$ for all $P \in \mathrm{Spec}^1(R)$ as desired. □

With that lemma in hand, we are able to establish the bijection through the map given. The proof is straightforward.

Theorem 3.3.2. *Let R be a Krull domain. Then the map from $\mathbb{Z}\{\mathrm{Spec}^1(R)\} \to \mathrm{Div}(R)$ defined by*

$$\mu \mapsto \{x \in R_0 : v_P(x) \geq \mu(P) \textit{ for all } P \in \mathrm{Spec}^1(R)\} \cup \{0\}$$

is a bijection.

Proof. We shall show that the map is well-defined, injective, and surjective.

Well-definition: Fix a function $\mu \in \mathbb{Z}\{\mathrm{Spec}^1(R)\}$. it is clear that the map defines a subset of R; We denote this by I. It only remains to show that this set is in fact a divisor of R. By the properties of valuations regarding addition and multiplication, it is clear that I is an R-submodule. To see that it is a fractional ideal, suppose that $\mu(P) < 0$ for $P = P_1, \dots, P_m$, $\mu(P) \geq 0$ for $P \neq P_1, \dots, P_m$. Take r_i to be a generator of $P_i R_{P_i}$ for each $i, 1 \leq i \leq m$ and let $r = \prod_{i=1}^{m} r_i^{-\mu(P_i)} \in R$. Then

$$\begin{aligned} rI &= \{rx \in R_0 : v_P(x) \geq \mu(P) \text{ for all } P \in \mathrm{Spec}^1(R)\} \cup r\{0\} \\ &= \{x \in R_0 : v_P\left(\frac{x}{r}\right) \geq \mu(P) \text{ for all } P \in \mathrm{Spec}^1(R)\} \cup \{0\} \\ &= \{x \in R_0 : v_P(x) \geq \mu(P) + v_P(r) \text{ for all } P \in \mathrm{Spec}^1(R)\} \cup \{0\}. \end{aligned}$$

Clearly $\mu(P) + v_P(r)$ is non-negative for all $P \in \mathrm{Spec}^1(R)$, so $rI \subseteq R$. Thus $I \in \mathrm{Id}(R)$ as desired. Finally,

$$\begin{aligned}
\operatorname{div}(I) &= \bigcap_{P\in\operatorname{Spec}^1(R)} IR_P \text{ (by Theorem 3.2.5)} \\
&= \bigcap_{P\in\operatorname{Spec}^1(R)} (PR_P)^{\mu(P)} \text{ (by Lemma 3.3.1)} \\
&= \bigcap_{P\in\operatorname{Spec}^1(R)} (\{x \in R_0 : v_P(x) \geq \mu(P)\} \cup \{0\}) \\
&= \{x \in R_0 : v_P(x) \geq \mu(P) \text{ for all } P \in \operatorname{Spec}^1(R)\} \cup \{0\} \\
&= I,
\end{aligned}$$

so I is a divisor as desired.

Injection: By Lemma 3.3.1, $IR_P = (PR_P)^{\mu(P)}$ for all $P \in \operatorname{Spec}^1(R)$; thus, the divisor I uniquely determines $\mu(P)$ for each $P \in \operatorname{Spec}^1(R)$, hence fixes μ as desired.

Surjection: For each divisor $I \in \operatorname{Div}(R)$, $I = \operatorname{div}(I) = \bigcap_{P\in\operatorname{Spec}^1(R)} IR_P$ by Lemma 3.3.1; for each $P \in \operatorname{Spec}^1(R)$, let r_P be a generator of PR_P. Then $IR_P = r_P^{k_P} R_P = (r_P R_P)^{k_P} = (PR_P)^{k_P}$ for some $k_P \in \mathbb{Z}$ by Lemma 3.2.2. Setting $\mu(P) = k_P$ for each $P \in \operatorname{Spec}^1(R)$ yields $IR_P = (PR_P)^{\mu(P)} = \{x \in R_0 : v_P(x) \geq \mu(P)\} \cup \{0\}$, whence $\mu \mapsto \bigcap_{P\in\operatorname{Spec}^1(R)} IR_P = I$. By Lemma 3.2.3, $IR_P = R_P$ for almost all $P \in \operatorname{Spec}^1(R)$, so μ is zero almost everywhere. Hence $\mu \in \mathbb{Z}\{\operatorname{Spec}^1(R)\}$ as desired. □

We denote an element of $\mathbb{Z}\{\operatorname{Spec}^1(R)\}$ corresponding to a divisor $I \in \operatorname{Div}(R)$ by μ_I.

Given the bijection between $\operatorname{Div}(R)$ and $\mathbb{Z}\{\operatorname{Spec}^1(R)\}$, the isomorphism follows easily. In the following theorem we fully prove this fact, as well as several tools which allow us to identify divisors through the bijection and information about their corresponding function μ.

Theorem 3.3.3. *Let R be a Krull domain. Then the map from $\operatorname{Div}(R) \to \mathbb{Z}\{\operatorname{Spec}^1(R)\}$ defined by $I \mapsto \mu_I$ is an isomorphism satisfying the following conditions:*

1. $\mu_{\operatorname{div}(IJ)} = \mu_I + \mu_J$
2. $I \subseteq J \;\Leftrightarrow\; \mu_I \geq \mu_J$
3. $\mu_{I:J} = \mu_I - \mu_J$
4. $\mu_{\operatorname{div}(I+J)} = min(\mu_I, \mu_J)$
5. $\mu_{I\cap J} = max(\mu_I, \mu_J)$

Proof. We know the map is a bijection by Theorem 3.3.2; it follows from item 1 that it is also a homomorphism, hence an isomorphism. Thus, it suffices to prove each of the items listed.

1. $\operatorname{div}(IJ) = \bigcap_{P\in\operatorname{Spec}^1(R)} (IJ)R_P$ by Theorem 3.2.5. Now, for each $P \in \operatorname{Spec}^1(R)$, we have that $(IJ)R_P = (IR_P)(JR_P) = (PR_P)^{\mu_I(P)}(PR_P)^{\mu_J(P)} = (PR_P)^{\mu_I(P)+\mu_J(P)} = (PR_P)^{(\mu_I+\mu_J)(P)}$, so in fact $\operatorname{div}(IJ) = \bigcap_{P\in\operatorname{Spec}^1(R)} (PR_P)^{(\mu_I+\mu_J)(P)}$. But we have already seen that $\operatorname{div}(IJ) = \bigcap_{P\in\operatorname{Spec}^1(R)} (PR_P)^{(\mu_{\operatorname{div}(IJ)})(P)}$, so by uniqueness we must have $\mu_{\operatorname{div}(IJ)}(P) = \mu_I + \mu_J(P)$ for all $P \in \operatorname{Spec}^1(R)$. It follows that $\mu_{\operatorname{div}(IJ)} = \mu_I + \mu_J$ as desired.

2. $I \subseteq J \Leftrightarrow IR_P \subseteq JR_P$ for all $P \in \operatorname{Spec}^1(R)$ (by Theorem 3.2.5) $\Leftrightarrow (PR_P)^{\mu_I(P)} \subseteq (PR_P)^{\mu_J(P)}$ for all $P \in \operatorname{Spec}^1(R) \Leftrightarrow \mu_I(P) \geq \mu_J(P)$ for all $P \in \operatorname{Spec}^1(R) \Leftrightarrow \mu_I \geq \mu_J$.

3. $x \in I : J \Leftrightarrow xJ \subseteq I \Leftrightarrow xJR_P \subseteq xIR_P$ for all $P \in \operatorname{Spec}^1(R)$ (by Theorem 3.2.5) $\Leftrightarrow v_P(x) + \mu_J(P) \geq \mu_I(P)$ for all $P \in \operatorname{Spec}^1(R) \Leftrightarrow v_P(x) \geq (\mu_I - \mu_J)(P)$ for all $P \in \operatorname{Spec}^1(R) \Leftrightarrow x \in (PR_P)^{(\mu_I-\mu_J)(P)}$ for all $P \in \operatorname{Spec}^1(R)$. This proves that $I : J = \bigcap_{P\in\operatorname{Spec}^1(R)} (PR_P)^{(\mu_I-\mu_J)(P)}$, whence $\mu_{I:J} = \mu_I - \mu_J$ as desired.

4. $\operatorname{div}(I+J) = \bigcap_{P\in\operatorname{Spec}^1(R)} (I+J)R_P$ (by Theorem 3.2.5) $= \bigcap_{P\in\operatorname{Spec}^1(R)} (IR_P + JR_P) = \bigcap_{P\in\operatorname{Spec}^1(R)} ((PR_P)^{\mu_I(P)} + (PR_P)^{\mu_J(P)}) = \bigcap_{P\in\operatorname{Spec}^1(R)} (PR_P)^{\min(\mu_I,\mu_J)(P)}$. By uniqueness, $\mu_{\operatorname{div}(I+J)} = \min(\mu_I, \mu_J)$ as desired.

5. Note that $IR_P \cap JR_P = (PR_P)^{\mu_I(P)} \cap (PR_P)^{\mu_J(P)} = (PR_P)^{\max(\mu_I,\mu_J)(P)}$ for all $P \in \operatorname{Spec}^1(R)$. Then $I \cap J = \left(\bigcap_{P\in\operatorname{Spec}^1(R)} IR_P\right) \cap \left(\bigcap_{P\in\operatorname{Spec}^1(R)} JR_P\right)$ (by Theorem 3.2.5) $= \bigcap_{P\in\operatorname{Spec}^1(R)} (IR_P \cap JR_P) = \bigcap_{P\in\operatorname{Spec}^1(R)} (PR_P)^{\max(\mu_I,\mu_J)(P)}$. By uniqueness, $\mu_{I\cap J} = \max(\mu_I, \mu_J)$ as desired.

□

Thus, our main result for the section is proven. We note that there is an intimate relationship between these values of μ and the valuations of the R_P; in particular, we have $\mu_{Rx}(P) = v_P(x)$ for all $P \in \operatorname{Spec}^1(R)$. If we denote the function $P \mapsto v_P(x)$ by $v(x)$, we can re-write this property as the equality $\mu_{Rx} = v(x)$.

3.4 The Approximation Theorem

In this section we prove one of the basic results pertaining to divisors in a Krull domain: The Approximation Theorem. Through it, we are able to prove the useful fact that every divisor is the intersection of two principal divisors, which will be very helpful to us in the sequel.

Theorem 3.4.1 (The Approximation Theorem). *Let R be a Krull domain, $P_1, \dots, P_m \in \operatorname{Spec}^1(R)$ distinct prime ideals, and $k_1, \dots, k_m$ integers. Then there exists a non-zero element $x \in R_0$ satisfying the following conditions:*

1. *$v_{P_i}(x) = k_i$ for each $i, 1 \leq i \leq m$, and*
2. *$v_P(x) \geq 0$ for all $P \in \operatorname{Spec}^1(R) \diagdown \{P_1, \dots, P_m\}$.*

Proof. By Lemma 3.2.1 we can find $\pi_1, \dots, \pi_m$ such that $v_{P_i}(\pi_j) = \delta_{ij}$ for $i, j, 1 \leq i, j \leq m$. Then $y = \prod_{i=1}^{m} \pi_i^{k_i}$ satisfies $v_{P_i}(y) = k_i$ for each $i, 1 \leq i \leq m$, as desired.

The problem thus far is that the values of the other v_P may be negative. To rectify this, let $P_{m+1}, \dots, P_n$ be the ideals $P \in \operatorname{Spec}^1(R) \diagdown \{P_1, \dots, P_m\}$ for which $v_P(y) < 0$ (recall that these are finite in number since $v_P(y) = 0$ for almost all $P \in \operatorname{Spec}^1(R)$). By Lemma 3.2.1 we can find $\pi_{m+1}, \dots, \pi_n$ such that $v_{P_i}(\pi_j) = \delta_{ij}$ for $i, j, 1 \leq i \leq n, m+1 \leq j \leq n$. Then $x = y \prod_{i=m+1}^{n} \pi_i^{-v_{P_i}(y)}$ still satisfies $v_{P_i}(y) = k_i$ for each $i, 1 \leq i \leq m$, and in addition $v_{P_i}(x) = 0$ for each $i, m+1 \leq i \leq n$. Finally, note that we have not introduced any more negative terms to the $P \in \operatorname{Spec}^1(R) \diagdown \{P_1, \dots, P_n\}$ because each $\pi_i \in R$ and each $-v_{P_i}(y) > 0$, so $\prod_{i=m+1}^{n} \pi_i^{-v_{P_i}(y)} \in R$ has a non-negative value for all $v_P, P \in \operatorname{Spec}^1(R)$. Therefore $v_P(x) \geq 0$ for all $P \in \operatorname{Spec}^1(R) \diagdown \{P_1, \dots, P_m\}$ as desired, and the result is proven. □

The Approximation Theorem allows us the following beautiful simplification in the definition of a divisor in a Krull domain, which allows us to be much more precise when we consider them in the future.

Theorem 3.4.2. *Let R be a Krull domain and $I \in \operatorname{Div}(R)$. Then I is the intersection of two principal divisors, i.e. $I = Rx_0 \cap Rx_1$ for some elements $x_0, x_1 \in R_0$.*

Proof. We first show that I is of the form $\operatorname{div}(Ry_0 + Ry_1)$ for some elements $y_0, y_1 \in R_0$. Let $\mu_I(P) = 0$ for $P \neq P_1, \dots, P_m$. By Theorem 3.4.1, we can find an element y_0 that satisfies

$$v_P(y_0) = \mu_I(P) \text{ for } P = P_1, \dots, P_m, \text{ and}$$

$$v_P(y_0) \geq 0 \text{ for all } P \in \operatorname{Spec}^1(R) \diagdown \{P_1, \dots, P_m\}.$$

Let $v_P(y_0) = 0$ for $P \neq P_{m+1}, \dots, P_n$. Applying the Approximation Theorem again, we can find an element y_1 that satisfies

$$v_P(y_1) = \mu_I(P) \text{ for } P = P_1, \dots, P_m,$$

$$v_P(y_1) = 0 \text{ for } P = P_{m+1}, \dots, P_n, \text{ and}$$

$$v_P(y_1) \geq 0 \text{ for all } P \in \operatorname{Spec}^1(R) \diagdown \{P_1, \dots, P_n\}.$$

With these constructions in place, it is clear that $\mu_I = \min(v(y_0), v(y_1)) = \min(\mu_{Ry_0}, \mu_{Ry_1})$. It follows from Theorem 3.3.3 that $I = \operatorname{div}(Ry_0 + Ry_1)$.

Clearly $R \in \text{Div}(R)$, so $R : I$ is a divisor by Proposition 3.1.1 item 2; by the result proven in the preceding part, we can find $y_0, y_1 \in R_0$ such that $R : I = \text{div}(Ry_0 + Ry_1)$. By Proposition 3.1.1 item 3 we can re-write this as $R : (R : (Ry_0 + Ry_1))$, whence $I = \text{div}(I) = \text{div}(R : (Ry_0 + Ry_1)) = R : (Ry_0 + Ry_1)$, where the equalities follow from Proposition 3.1.1 items 1, 4, and 2, respectively. Now we have

$$\begin{aligned} I &= R : (Ry_0 + Ry_1) \\ &= (R : Ry_0) \cap (R : Ry_1) \\ &= Ry_0^{-1} \cap Ry_1^{-1}, \end{aligned}$$

so setting $x_0 = y_0^{-1} \in R_0$ and $x_1 = y_1^{-1} \in R_0$ yields $I = Rx_0 \cap Rx_1$ as desired. □

Thus, our main result for the section is proven. Note that in the case of a Dedekind domain, where every fractional ideal is a divisor, we have re-proven the familiar result that every fractional ideal is generated by two elements.

This concludes our study of the fundamental properties of divisors. Our next task is to define and study an important invariant of Krull rings, the divisor class group.

Chapter 4

The Divisor Class Group

In this chapter we define the divisor class group and determine methods with which we can identify its structure. The divisor class group is important to us because we can use it as a measure of the uniqueness of factorization that exists in a Krull domain. After demonstrating this fact, the natural question to ask is how to calculate the class group of a given Krull domain. Our search for answers begins with a search for induced homomorphisms of ring extensions. After finding a small but useful class of extensions which induce homomorphisms of the divisor class groups, we prove Nagata's Theorem and a related result, each of which describe the kernel of a homomorphism induced from a certain type of extension. These results allow us to prove, among other things, that the class group a Krull domain is isomorphic to that of any polynomial extension ring, and to that of a certain class of localizations. These results will be the bread and butter of the constructions we perform in the final chapter.

4.1 Definition and Basic Properties

In this section, we define the semigroup of divisor classes for a general domain R. In the case of Krull domains, this will turn out to be the group which we call the divisor class group.

Given a domain R, we define the relation $\equiv$ on the semigroup of divisors $\mathrm{Div}(R)$ as follows:

$$I \equiv J \;\Leftrightarrow\; I = J + K \text{ for some } K \in \mathrm{Prin}(R),$$

where $I, J \in \mathrm{Div}(R)$. Since $J + K = J + Rx = \mathrm{div}(J(Rx)) = \mathrm{div}(xJ) = x\mathrm{div}(J) = xJ$ for some $x \in R_0 \setminus 0$, it is clear that this relation is reflexive, symmetric, and transitive, and hence an equivalence relation. Furthermore, if we have $I \equiv J$ and $I' \equiv J'$, then $I = xJ$

and $I' = x'J'$ for some $x, x' \in R_0 \setminus 0$, so

$$\begin{aligned} I + I' &= \operatorname{div}(II') \\ &= \operatorname{div}((xJ)(x'J')) \\ &= \operatorname{div}(xx'(JJ')) \\ &= xx'\operatorname{div}(JJ') \\ &= xx'(J + J'), \end{aligned}$$

where $xx' \in R_0 \setminus 0$; it follows that $I + I' \equiv J + J'$. Thus $\equiv$ is in fact a congruence relation, and is known as the **relation of linear equivalence of divisors**.

It also follows from the compatibility of the semigroup operation that this congruence relation is compatible with the map div : $\operatorname{Id}(R) \to \operatorname{Div}(R)$, i.e. if $\operatorname{div}(I) \equiv \operatorname{div}(J)$ and $\operatorname{div}(I') \equiv \operatorname{div}(J')$ then $\operatorname{div}(I) + \operatorname{div}(J) \equiv \operatorname{div}(I') + \operatorname{div}(J')$, where $I, J, I', J' \in \operatorname{Id}(R)$.

We can now define the semigroup of divisor classes.

Definition 4.1.1. The **semigroup of divisor classes** of a domain R is the quotient semigroup $\operatorname{Div}(R)/\equiv$.

We denote the semigroup of divisor classes of a domain R by $\operatorname{Cl}(R)$. Note that by Propostion 3.1.2, $\operatorname{Cl}(R)$ has the structure of a group whenever R is a completely normal domain; in particular, it is true whenever R is a Krull domain. In these cases, the semigroup of divisor classes is renamed the **divisor class group**, and satisfies $\operatorname{Cl}(R) = \operatorname{Div}(R)/\operatorname{Prin}(R)$.

Notice that in the case of Dedekind domains, every non-zero fractional ideal is a divisor. Thus $\operatorname{Cl}(R) = \operatorname{Div}(R)/\operatorname{Prin}(R) = \operatorname{Id}(R)/\operatorname{Prin}(R)$, so this definition coincides with that of the ideal class group, which is also denoted by $\operatorname{Cl}(R)$.

4.2 Measurement of Unique Factorization

Here we present the theorem which is one of the fundamental reasons the divisor class group is of so much interest. Because a Krull domain has unique factorization if and only if its divisor class group is trivial, we can consider the size of the divisor class group to be a measure of how far we are from a unique factorization domain. The larger $\operatorname{Cl}(R)$, the less unique its factorization. The phenomenon that occurs here is identical to that studied for the ideal class group of a Dedekind domain, but generalized to include a much wider class of rings.

Theorem 4.2.1. *Let R be a ring. Then the following conditions are equivalent:*

1. *R is a unique factorization domain.*
2. *R is a Krull domain with* $\operatorname{Cl}(R) = 0$.

PROOF. We show that each condition implies the other.

$1 \Rightarrow 2$: Let R be a unique factorization domain. We have from Theorem 2.3.1 that R is a Krull domain, and that every divisor $P \in \operatorname{Spec}^1(R)$ is principal. Since the height 1 prime ideals span $\mathrm{Div}(R)$ by Theorem 3.3.3, we have that every divisor is the sum of principal divisors. But sums of principal divisors $Rx + Ry = \mathrm{div}((Rx)(Ry)) = \mathrm{div}(Rxy) = Rxy$ are again principal divisors, so in fact all divisors are principal, i.e. $\mathrm{Div}(R) = \mathrm{Prin}(R)$. It follows that $\mathrm{Cl}(R) = \mathrm{Div}(R)/\mathrm{Prin}(R) = 0$ as desired.

$2 \Rightarrow 1$: Let R be a Krull domain with $\mathrm{Cl}(R) = 0$. Then $\mathrm{Div}(R)/\mathrm{Prin}(R) = \mathrm{Cl}(R) = 0$, so $\mathrm{Div}(R) = \mathrm{Prin}(R)$, whence all divisors are principal. Now, for each prime ideal $P \in \mathrm{Spec}^1(R)$ we take a generator π_P. In the localization R_P, π_P also generates the ideal $\pi_P R_P = \pi_P(RR_P) = (\pi_P R)R_P = PR_P$, the unique maximal ideal; thus $v_P(\pi_P) = 1$. On the other hand, $\pi_P \in P'$ for some $P' \in \mathrm{Spec}^1(R) \Rightarrow P = \pi_P R \subseteq P'$, whence $P' = P$ since it is height 1. Thus, $\pi_P \notin P'$ so that $v_{P'}(\pi_P) = 0$ for any $P' \in \mathrm{Spec}^1(R), P' \neq P$.

If we write $\pi_P = ab$ for some $a, b \in R$, then $v'_P(a), v'_P(b) \geq 0$ for all $P' \in \mathrm{Spec}^1(R)$, but $v'_P(a) + v'_P(b) = v'_P(ab)$ is 1 for $P' = P$, 0 otherwise. Thus one of $v_P(a)$ and $v_P(b)$ is 0, the other is 1, and all the other $v'_P(a), v'_P(b)$ are 0. Thus, either $v'_P(a) = 0$ for all $P' \in \mathrm{Spec}^1(R)$, or $v'_P(b) = 0$ for all $P' \in \mathrm{Spec}^1(R)$, whence either a or b is invertible in R. It follows that each of the π_P are irreducible elements of R.

For each non-zero element $r \in R$, let $r' = \prod_{P \in \mathrm{Spec}^1(R)} \pi_P^{v_P(r)}$. Then

$$\begin{aligned}
v_{P'}(\frac{r}{r'}) &= v_{P'}(r) - v_P(r') \\
&= v_{P'}(r) - \left(\sum_{P \in \mathrm{Spec}^1(R)} v_P(r) v_{P'}(\pi_P) \right) \\
&= v_{P'}(r) - v'_P(r) \\
&= 0
\end{aligned}$$

for every $P' \in \mathrm{Spec}^1(R)$, so $u = \frac{r}{r'}$ is an invertible element of R and

$$r = ur' = u \left(\prod_{P \in \mathrm{Spec}^1(R)} \pi_P^{v_P(r)} \right).$$

This gives us factorization in R.

Now, since every non-zero element of R can be expressed as the unit multiple of a product of these π_P, they and their associates are the only irreducibles in the ring. If a non-zero element $r \in R$ can be factored as

$$r = u \left(\prod_{P \in \mathrm{Spec}^1(R)} \pi_P^{\alpha_P} \right),$$

then

$$v_{P'}(r) = v_{P'}\left(u \left(\prod_{P \in \mathrm{Spec}^1(R)} \pi_P^{\alpha_P} \right) \right) = v_{P'}(u) + \left(\sum_{P \in \mathrm{Spec}^1(R)} \alpha_P v_{P'}(\pi_P) \right) = \alpha_{P'},$$

so that the factorization is exactly the one given above. Thus, the factorization is unique up to units, whence R is a unique factorization domain as desired. $\square$

4.3 Induced Homomorphisms of Class Groups

In this section, we study homomorphisms between divisor class groups of related rings. Ideally, we would like to have a situation where any extension of Krull domains induces a homomorphism of their class groups. Unfortunately, this is not true in general, and happens on a relatively small class of Krull domain extensions. Nevertheless, these extensions are special enough that they will be of great use to us in the future.

Given $j : R \hookrightarrow R'$ an extension of Krull domains, we can define a map of fractional ideals $\mathrm{Id}(j) : \mathrm{Id}(R) \to \mathrm{Id}(R')$ by

$$\mathrm{Id}(j)(I) = IR'$$

for every $I \in \mathrm{Id}(R)$. Unfortunately, this does not carry divisors to divisors. Instead, we define an alternative map $\mathrm{Div}(j) : \mathrm{Div}(R) \to \mathrm{Div}(R')$ that takes divisors to divisors in the natural way, i.e.

$$\mathrm{Div}(j)(I) = \mathrm{div}(IR').$$

Although this map is not always a homomorphism, it turns out that it has enough nice properties that it is useful to us. In particular, it turns out to be a homomorphism for certain special classes of extensions, which we describe later.

Note that the image of any principal divisor Rx is $\mathrm{Div}(j)(Rx) = \mathrm{div}((Rx)R') = \mathrm{div}(R'x) = R'x$, a principal divisor of R'. Thus, $\mathrm{Div}(j)$ carries principal divisors into principal divisors, i.e. $\mathrm{Div}(j)(\mathrm{Prin}(R)) \subseteq \mathrm{Prin}(R')$.

In preparation for our investigation as to which extensions admit an induced homomorphism, we prove a few useful lemmas.

Lemma 4.3.1. *Let $j : R \hookrightarrow R'$ be an extension of Krull domains, $P' \in \mathrm{Spec}^1(R')$, $P = P' \cap R \in Spec(R)$. If $\mathrm{ht}(P) \leq 1$ then there exists a constant $e(P'/P)$ such that*

$$\mu_{\mathrm{div}(IR')}(P') = e(P'/P)\mu_I(P)$$

for each divisor $I \in \mathrm{Div}(R)$. Moreover, if $\mathrm{ht}(P) = 0$, then $e(P'/P) = 0$.

Proof. $R \setminus P \subseteq R' \setminus P'$, so $R_P \subseteq R'_{P'}$, whence $R_P R'_{P'} = R'_{P'}$. Then $(P'R'_{P'})^{\mu'(P')} = (IR')R'_{P'}$ (by Theorem 3.3.1) $= IR'_{P'} = (IR_P)R'_{P'}$.

If $\mathrm{ht}(P) = 0$ then $P = 0$, i.e. R intersects P' trivially. Then every non-zero element of IR_P is invertible in $R'_{P'}$, so that $(P'R'_{P'})^{\mu'(P')} = (IR_P)R'_{P'} = R'_{P'}$; it follows that $\mu'(P') = 0$, so we set $e(P'/P) = 0$ as desired and the result follows.

If $\mathrm{ht}(P) = 1$ then P, and hence PR_P, is non-zero. Then $(PR_P)R'_{P'}$ is a non-zero ideal of $R'_{P'}$, so by Lemma 3.2.2 there exists a unique integer $e(P'/P)$ such that $(PR_P)R'_{P'} = (P'R'_{P'})^{e(P'/P)}$. This satisfies

$$\begin{aligned}
(P'R'_{P'})^{\mu'(P')} &= (IR_P)R'_{P'} \text{ (from above)} \\
&= (PR_P)^{\mu(P)}R'_{P'} \text{ (by Lemma 3.3.1)} \\
&= (PR_PR'_{P'})^{\mu(P)} \\
&= (P'R'_{P'})^{e(P'/P)\mu(P)}
\end{aligned}$$

(raising both sides of the previous equation to the power of $\mu(P)$). Therefore $\mu'(P') = e(P'/P)\mu(P)$ as desired. ∎

Corollary 4.3.2. *Let $j : R \hookrightarrow R'$ be an extension of Krull domains. If the height of the prime ideal $P = P' \cap R$ is less than or equal to 1 for every ideal $P' \in \operatorname{Spec}^1(R')$, then*

$$\operatorname{Div}(j)(P) = \sum_{P' \in \operatorname{Spec}^1(R'), P' \cap R = P} e(P'/P)P'.$$

Lemma 4.3.3. *Let $j : R \hookrightarrow R'$ be an extension of Krull domains. If the height of the prime ideal $P = P' \cap R$ is less than or equal to 1 for every ideal $P' \in \operatorname{Spec}^1(R')$, then the map $\operatorname{Div}(j) : \operatorname{Div}(R) \to \operatorname{Div}(R')$ taking $I \mapsto \operatorname{div}(IR')$ is a group homomorphism.*

PROOF. We may view $\operatorname{Div}(j)$ as a function taking elements of $\mathbb{Z}\{\operatorname{Spec}^1(R)\}$ to elements of $\mathbb{Z}\{\operatorname{Spec}^1(R')\}$ since these groups are isomorphic to $\operatorname{Div}(R)$ and $\operatorname{Div}(R')$ by Theorem 3.3.3. Fix $\mu_1, \mu_2 \in \mathbb{Z}\{\operatorname{Spec}^1(R)\}$. Then for all prime ideals $P' \in \operatorname{Spec}^1(R')$, $P = P' \cap R$, we have by Lemma 4.3.1 that

$$\begin{aligned}
(\operatorname{Div}(j)(\mu_1 + \mu_2))(P') &= e(P'/P)((\mu_1 + \mu_2)(P)) \\
&= e(P'/P)\mu_1(P) + e(P'/P)\mu_2(P) \\
&= (\operatorname{Div}(j)(\mu_1))(P') + (\operatorname{Div}(j)(\mu_2))(P').
\end{aligned}$$

It follows that $\operatorname{Div}(j)(\mu_1+\mu_2) = \operatorname{Div}(j)(\mu_1)+\operatorname{Div}(j)(\mu_2)$, whence $\operatorname{Div}(j)$ is a homomorphism as claimed. ☐

Lemma 4.3.4. *Let $R \hookrightarrow R'$ be a flat extension of rings and $I, J \in \operatorname{Id}(R)$. Then $R'(I \cap J) = R'I \cap R'J$.*

PROOF. Suppose $I, J \subseteq R$. Then

$$0 \to I \cap J \to R \to R/I \oplus R/J$$

is an exact sequence, where the map taking $R \to R/I \oplus R/J$ is the natural projection defined by $r \mapsto (r + I, r + J)$. Since R' is a flat R-module, tensoring with R' yields a new exact sequence

$$0 \otimes_R R' \to (I \cap J) \otimes_R R' \to R \otimes_R R' \to (R/I \oplus R/J) \otimes_R R',$$

i.e.

$$0 \to (I \cap J)R' \to R' \to R'/IR' \oplus R'/JR',$$

where the map taking $R' \to R'/IR' \oplus R'/JR'$ is the natural projection defined by $r' \mapsto (r' + IR', r' + JR')$. Then by exactness, the kernel of this map is $(I \cap J)R'$; by inspection, it is also $IR' \cap JR'$. Thus, $(I \cap J)R' = IR' \cap JR'$, and the result follows.

Suppose $I, J \nsubseteq R$. Then we can find $r_1, r_2 \in R$ such that $r_1 I, r_2 J \subseteq R$, whence $r_1 r_2 I, r_1 r_2 J \subseteq R$. By the preceding result, $R'(r_1 r_2 I \cap r_1 r_2 J) = R'(r_1 r_2 I) \cap R'(r_1 r_2 J)$, or equivalently, $r_1 r_2 R'(I \cap J) = r_1 r_2 (R'I \cap R'J)$. It follows that $R'(I \cap J) = R'I \cap R'J$ as desired. ☐

Corollary 4.3.5. *Let $R \hookrightarrow R'$ be a flat extension of Krull domains and $I \in \operatorname{Div}(R)$. Then $R'I \in \operatorname{Div}(R')$.*

PROOF. By Theorem 3.4.2, we have $I = Rx_1 \cap Rx_2$. Then $R'I = R'(Rx_1 \cap Rx_2) = R'Rx_1 \cap R'Rx_2$ (by Lemma 4.3.4) $= R'x_1 \cap R'x_2$, and hence is a divisor as claimed. $\square$

Lemma 4.3.6. *Let $R \hookrightarrow R'$ be a flat extension of Krull domains. Then the height of the prime ideal $P = P' \cap R$ is less than or equal to 1 for every ideal $P' \in \mathrm{Spec}^1(R')$.*

PROOF. Suppose there exists $P_0' \in \mathrm{Spec}^1(R')$ such that $\mathrm{ht}(P_0' \cap R) > 1$; choose non-zero $r \in P_0' \cap R$. Then r is contained in only finitely many members of $\mathrm{Spec}^1(R)$, say $P_1, \dots, P_m$. Choose $s \in (P_0' \cap R) \setminus \bigcup_{i=1}^{m} P_i$ (this must be non-empty since otherwise $P_0' \cap R \subseteq P_i$ for some $i, 1 \leq i \leq m$, so that $\mathrm{ht}(P_i) > 1$ as well). Now, $r, s \in R \Rightarrow v_P(r), v_P(s) \geq 0$ for all $P \in \mathrm{Spec}^1(R)$, and

$$v_p(r) = 0 \text{ for } P \neq P_1, \dots P_m \text{ while}$$

$$v_p(s) = 0 \text{ for } P = P_1, \dots P_m.$$

It follows that

$$v_P(rs) = v_P(s) = \max(v_P(r), v_P(s)) \text{ for } P \neq P_1, \dots P_m \text{ while}$$

$$v_P(rs) = v_P(r) = \max(v_P(r), v_P(s)) \text{ for } P = P_1, \dots P_m,$$

whence $v_P(rs) = \max(v_P(r), v_P(s))$ for all $P \in \mathrm{Spec}^1(R)$, i.e. $\mu_{Rrs} = \max(\mu_{Rr}, \mu Rs)$. Then by Theorem 3.3.3, we have $Rrs = Rr \cap Rs$.

By Lemma 4.3.4, $R'(Rr \cap Rs) = R'(Rr) \cap R'(Rs) = R'r \cap R's$, so $R'rs = R'(Rrs) = R'(Rr \cap Rs)$ (from above) $= R'r \cap R's$. Then by Theorem 3.3.3, we have $v_{P'}(rs) = \max(v_{P'}(r), v_{P'}(s))$ for all $P' \in \mathrm{Spec}^1(R')$ as well. But $r, s \in P_0'$ so $v_{P_0'}(rs) = v_{P_0'}(r) + v_{P_0'}(s) > \max(v_{P_0'}(r), v_{P_0'}(s))$, a contradiction. Thus, we must have that $\mathrm{ht}(P_0' \cap R) \leq 1$ for all $P' \in \mathrm{Spec}^1(R')$ as desired. $\square$

Theorem 4.3.7. *Let $j : R \hookrightarrow R'$ be an extension of Krull domains satisfying one of the following conditions:*

1. *R' is a flat R-module.*
2. *R' is a localization of R.*
3. *R' is a subintersection of R.*
4. *R' is integral over R.*

Then $\mathrm{Div}(j)$ *and* $\mathrm{Cl}(j)$ *are group homomorphisms.*

PROOF. It is sufficient to show for each case that the height of the prime ideal $P = P' \cap R$ is less than or equal to 1 for every ideal $P' \in \mathrm{Spec}^1(R')$; it then follows from Lemma 4.3.3 that $\mathrm{Div}(j)$ is a homomorphism. Since $\mathrm{Div}(j)$ sends principal divisors to principal divisors, this induces the homomorphism $\mathrm{Cl}(j)$ as well.

1. This is the result proven in Lemma 4.3.6.
2. This follows from the preceding part since localizations are flat R-modules.

3. It is easy to see that $\mathrm{Spec}^1(R') = \mathrm{Spec}^1(R(\Lambda)) = \Lambda \subseteq \mathrm{Spec}^1(R)$, so the result follows.

4. This follows directly from the Going Down Theorem; if $P' \cap R$ contains a distinct prime ideal, then so does P'. Since P' is of height 1, it contains no other prime ideals, so neither does $P = P' \cap R$; thus $\mathrm{ht}(P) \leq 1$. Since this holds for all $P' \in \mathrm{Spec}^1(R')$, the result follows.

$\square$

4.4 Nagata's Theorem

Now that we have families of extensions $R \hookrightarrow R'$ for which homomorphisms exist between $\mathrm{Cl}(R)$ and $\mathrm{Cl}(R')$, we can begin to use them to determine relationships between the divisor class groups of related rings. Our first result is a powerful theorem due to Nagata. The proof first appeared in Nagata [6].

Theorem 4.4.1 (Nagata's Theorem). *Let R be a Krull domain and R_S a localization of R at some multiplicative subset $S \subseteq R$. Then*

$$F \xrightarrow{f} \mathrm{Cl}(R) \xrightarrow{\mathrm{Cl}(j)} \mathrm{Cl}(R_S) \longrightarrow 0$$

is an exact sequence, where F is the subgroup of $\mathrm{Div}(R)$ *generated by the set* $\{P \in \mathrm{Spec}^1(R) : P \cap S \neq \emptyset\}$, *$f : F \to \mathrm{Cl}(R)$ is the composition $F \hookrightarrow \mathrm{Div}(R) \to \mathrm{Cl}(R)$, and $\mathrm{Cl}(j) : \mathrm{Cl}(R) \to \mathrm{Cl}(R_S)$ is the homomorphism of class groups induced by $j : R \to R_S$. Moreover,*

$$\mathrm{Ker}(\mathrm{Cl}(j)) = (F + \mathrm{Prin}(R))/\mathrm{Prin}(R).$$

Proof. First note that the induced homomorphism $\mathrm{Cl}(j)$ is valid by Theorem 4.3.7. Now, let $\Lambda = \{P \in \mathrm{Spec}^1(R) : P \cap S \neq \emptyset\}$ so that $F = \mathbb{Z}\{\Lambda\}$. Recalling that $\mathrm{Spec}^1(R_S) = \mathrm{Spec}^1(R) \diagdown \Lambda$ and using the identifications of Theorem 3.3.3 $\mathrm{Div}(R) \cong \mathbb{Z}\{\mathrm{Spec}^1(R)\}$ and $\mathrm{Div}(R_S) \cong \mathbb{Z}\{\mathrm{Spec}^1(R_S)\} = \mathbb{Z}\{\mathrm{Spec}^1(R) \diagdown \Lambda\}$, we can make the further identification $\mathrm{Div}(R) \cong \mathrm{Div}(R_S) \oplus F$.

Now, consider the function of $\mathrm{Div}(j)$ on the basis elements $P \in \mathrm{Spec}^1(R)$ of $\mathrm{Div}(R)$. The image is either the corresponding basis element in $\mathrm{Div}(R_S)$ (in the case that $P \notin \Lambda$), or the whole ring R_S (in the case that $P \in \Lambda$); thus the image of $\mathrm{Div}(j)$ is exactly the component $\mathrm{Div}(R_S)$ (the first summand), so the image of $\mathrm{Cl}(j)$ is the entirety of $\mathrm{Cl}(R_S)$, whence $\mathrm{Cl}(j)$ is surjective. Also note that $\mathrm{Ker}(\mathrm{Div}(j))$ is then exactly F.

Finally, observe that for each principal divisor $\frac{a}{b}R_S \in \mathrm{Prin}(R_S)$, $a \in R, b \in S \subseteq R$, there is a principal divisor $\frac{a}{b}R \in \mathrm{Prin}(R)$ such that $\mathrm{Div}(j)(\frac{a}{b}R) = \mathrm{div}((\frac{a}{b}R)R_S) = \mathrm{div}(\frac{a}{b}R_S) = \frac{a}{b}R_S$, whence $\mathrm{Prin}(R_S) \subseteq \mathrm{Div}(j)(\mathrm{Prin}(R))$. Combining this with the established fact $\mathrm{Div}(j)(\mathrm{Prin}(R)) \subseteq \mathrm{Prin}(R_S)$ yields $\mathrm{Div}(j)(\mathrm{Prin}(R)) = \mathrm{Prin}(R_S)$, so that $\mathrm{Ker}(\mathrm{Cl}(j)) = (\mathrm{Ker}(\mathrm{Div}(j)) + \mathrm{Prin}(R))/\mathrm{Prin}(R) = (F + \mathrm{Prin}(R))/\mathrm{Prin}(R)$ as desired. This last is also clearly $\mathrm{Im}(f)$, completing the proof of exactness. $\square$

The crucial concept behind Nagata's Theorem is the identification of $\mathrm{Div}(R)$ with $\mathrm{Div}(R_S) \oplus F$. Using this idea allows us to come up with a similar result for subintersections.

Theorem 4.4.2. *Let R be a Krull domain and $R(\Lambda)$ a subintersection of R for some subset $\Lambda \subseteq \mathrm{Spec}^1(R)$. Then*

$$F \xrightarrow{f} \mathrm{Cl}(R) \xrightarrow{\mathrm{Cl}(j)} \mathrm{Cl}(R(\Lambda)) \longrightarrow 0$$

is an exact sequence, where F is the subgroup of $\mathrm{Div}(R)$ generated by the set $\mathrm{Spec}^1(R)\setminus\Lambda$, $f : F \to \mathrm{Cl}(R)$ is the composition $F \hookrightarrow \mathrm{Div}(R) \to \mathrm{Cl}(R)$, and $\mathrm{Cl}(j) : \mathrm{Cl}(R) \to \mathrm{Cl}(R(\Lambda))$ is the homomorphism of class groups induced by $j : R \to R(\Lambda)$. Moreover,

$$\mathrm{Ker}(\mathrm{Cl}(j)) = (F + \mathrm{Prin}(R))/\mathrm{Prin}(R).$$

Proof. First note that the induced homomorphism $\mathrm{Cl}(j)$ is valid by Theorem 4.3.7. We have $F = \mathbb{Z}\{\mathrm{Spec}^1(R)\setminus\Lambda\}$. Recalling that $\mathrm{Spec}^1(R(\Lambda)) = \Lambda$ and using the identifications of Theorem 3.3.3 $\mathrm{Div}(R) \cong \mathbb{Z}\{\mathrm{Spec}^1(R)\}$ and $\mathrm{Div}(R(\Lambda)) \cong \mathbb{Z}\{\mathrm{Spec}^1(R(\Lambda))\} = \mathbb{Z}\{\Lambda\}$, we can make the further identification $\mathrm{Div}(R) \cong \mathrm{Div}(R(\Lambda)) \oplus F$.

Now, consider the function of $\mathrm{Div}(j)$ on the basis elements $P \in \mathrm{Spec}^1(R)$ of $\mathrm{Div}(R)$. The image is either the corresponding basis element in $\mathrm{Div}(R(\Lambda))$ (in the case that $P \in \Lambda$), or the whole ring $R(\Lambda)$ (in the case that $P \notin \Lambda$); thus the image of $\mathrm{Div}(j)$ is exactly the component $\mathrm{Div}(R(\Lambda))$ (the first summand), so the image of $\mathrm{Cl}(j)$ is the entirety of $\mathrm{Cl}(R(\Lambda))$, whence $\mathrm{Cl}(j)$ is surjective. Also note that $\mathrm{Ker}(\mathrm{Div}(j))$ is then exactly F.

Finally, we again have $\mathrm{Spec}^1(R(\Lambda)) \subseteq \mathrm{Spec}^1(R)$. Since $\mathrm{Div}(j)$ is a projection onto the $\mathbb{Z}\{\Lambda\}$ summand, $\mathrm{Prin}(R(\Lambda)) \subseteq \mathrm{Div}(j)(\mathrm{Prin}(R))$ and thus $\mathrm{Div}(j)(\mathrm{Prin}(R)) = \mathrm{Prin}(R(\Lambda))$, so that $\mathrm{Ker}(\mathrm{Cl}(j)) = (\mathrm{Ker}(\mathrm{Div}(j))+\mathrm{Prin}(R))/\mathrm{Prin}(R) = (F+\mathrm{Prin}(R))/\mathrm{Prin}(R)$ as desired. This last is also clearly $\mathrm{Im}(f)$, completing the proof of exactness. □

With these two powerful theorems in hand, we can go forth to find even more useful results: isomorphisms of divisor class groups.

4.5 Class Group Isomorphisms

In this section, we finally present two theorems that represent the fruits of our studies of induced homomorphisms. Each one gives us an isomorphism between a domain R and an extension ring R', which is something that will be very helpful to us when we begin to construct rings with specified divisor class groups.

We begin with a theorem on localizations which is a direct result of Nagata's Theorem.

Theorem 4.5.1. *Let R be a Krull domain and $\{p_\gamma\}_{\gamma\in\Gamma}$ a set of elements in R such that Rp_γ is a prime ideal of R for each $\gamma \in \Gamma$. Let S be the multiplicative subset of R consisting of all finite products of the p_γ. Then the homomorphism $\mathrm{Cl}(j)$ induced by the extension $j : R \hookrightarrow R_S$ is an isomorphism.*

Proof. We have that Cl(j) is a well-defined homomorphism from Theorem 4.3.7, and surjective from Nagata's Theorem. Thus, it only remains to show that it is injective.

For $P \in \mathrm{Spec}^1(R)$, $P \cap S \neq \emptyset \Rightarrow \exists s = \prod p_\gamma \in P$, so by primality there is some $p_\gamma \in P$. Then the prime ideal $Rp_\gamma \subseteq P$, but $\mathrm{ht}(P) = 1$ so we must have that $P = Rp_\gamma \in \mathrm{Prin}(R)$. By Nagata's Theorem, we have that the basis for F is $\{P \in \mathrm{Spec}^1(R) : P \cap S \neq \emptyset\} \subseteq \mathrm{Prin}(R)$, so $F \subseteq \mathrm{Prin}(R)$. It follows that $\mathrm{Ker}(\mathrm{Cl}(j)) = (F+\mathrm{Prin}(R))/\mathrm{Prin}(R) = \mathrm{Prin}(R)/\mathrm{Prin}(R) = 0$, whence Cl(j) is injective as desired. □

Next, we present a more difficult theorem for polynomial extension rings.

Theorem 4.5.2. *Let R be a Krull domain and $\{X_\gamma\}_{\gamma\in\Gamma}$ a set of indeterminates. Then the homomorphism* $\mathrm{Cl}(j)$ *induced by the extension* $j : R \hookrightarrow R[\{X_\gamma\}_{\gamma\in\Gamma}]$ *is an isomorphism.*

PROOF. We have that Cl(j) is a well-defined homomorphism from Theorem 4.3.7. Thus, it only remains to show that it is injective and surjective. Let $T = R[\{X_\gamma\}_{\gamma\in\Gamma}]$.

Injection: We compute $\mathrm{Ker}(\mathrm{Cl}(j))$. Suppose $I + \mathrm{Prin}(R) \mapsto \mathrm{Prin}(T)$. Take $r \in R : I$ and let $J = rI \subseteq R$ so that $J+\mathrm{Prin}(R) = I+\mathrm{Prin}(R) \mapsto \mathrm{Prin}(T)$, i.e. $\mathrm{Div}(j)(J) \in \mathrm{Prin}(T)$. By Corollary 4.3.5 JT is a divisor, so $\mathrm{Div}(j)(J) = \mathrm{div}(JT) = JT = Tt$ for some $t \in T$. Now, J and T both intersect R, so $Tt = JT$ has some constant elements (i.e. elements of R); thus t must be constant as well. Then $J = JT \cap R = Tt \cap R = Rt \in \mathrm{Prin}(R)$, so $I + \mathrm{Prin}(R) = J + \mathrm{Prin}(R) = \mathrm{Prin}(R)$. Thus $\mathrm{Ker}(\mathrm{Cl}(j)) = 0$, whence $\mathrm{Cl}(j)$ is injective as desired.

Surjection: We begin by determining $\mathrm{Cl}(T)$. Consider the inclusion $i : T \hookrightarrow T_{R\setminus 0}$; by Theorem 4.3.7 this induces a homomorphism $\mathrm{Cl}(T) \to \mathrm{Cl}(T_{R\setminus 0})$, and we can apply Theorem 4.2.1 to see that in fact $\mathrm{Cl}(T_{R\setminus 0}) = 0$ (since $T_{R\setminus 0} = R_0[\{X_\gamma\}_{\gamma\in\Gamma}]$ is a unique factorization domain). Thus $\mathrm{Ker}(\mathrm{Cl}(i)) = \mathrm{Cl}(T)$. But Theorem 4.4.1 gives us an alternate description of the kernel, whereby setting $F \subseteq \mathrm{Div}(T)$ as the free $\mathbb{Z}$-module with basis $\{Q \in \mathrm{Spec}^1(T) : Q \cap R \neq 0\}$ gives us $\mathrm{Cl}(T) = \mathrm{Ker}(\mathrm{Cl}(i)) = (F + \mathrm{Prin}(T))/\mathrm{Prin}(T)$.

Recall from Proposition 2.2.4 that each prime ideal $Q \in \mathrm{Spec}^1(T)$ is of one of the two forms

1. $Q = R_0[\Gamma]q \cap T, q \in R_0[\Gamma]$ irreducible (in the case that $Q \cap R = 0$) or
2. $Q = (Q \cap R)T, Q \cap R \in \mathrm{Spec}^1(R)$ (in the case that $Q \cap R \neq 0$).

By the values of the intersections with R, we see that those of form 2 are exactly the basis elements of F; moreover, $\mathrm{Div}(j) : P \mapsto PT$ for $P \in \mathrm{Spec}^1(R)$, so $\mathrm{Im}(\mathrm{Div}(j)) = F$. Then $\mathrm{Im}(\mathrm{Cl}(j)) = (F + \mathrm{Prin}(R))/\mathrm{Prin}(R) = \mathrm{Cl}(T)$, so $\mathrm{Cl}(j)$ is surjective as desired. ☐

This completes our study of the fundamental properties of divisor class groups.

Chapter 5

Constructions of Rings with Specified Class Groups

The goal of this chapter is to prove Claborn's Theorem, the culmination of our study of divisors in Krull domains. It states that every abelian group occurs as the ideal class group of a Dedekind domain.

The approach we take is to perform a construction in three parts, each a step that brings us closer to the result. Viewing an abelian group G as the quotient group $\mathbb{Z}\{\Gamma\}/H$ of a free abelian group on some set Γ, we successively construct

1. A Krull domain with divisor class group $\mathbb{Z}\{\Gamma\}$,
2. A Krull domain with divisor class group G, and
3. A Dedekind domain with divisor class group G.

This method is due to Claborn, who first published the proof in the article Claborn [2] in 1966. Mimicking his proof, we prove that each of the three constructions can be done one at a time in the following three sections. The final proof is given in the last section.

5.1 Free $\mathbb{Z}$-modules

In this section we demonstrate Step 1 of the construction. We begin with a lemma.

Lemma 5.1.1. *Let K be a field, $K[\Gamma] = K[\{X_\gamma\}_{\gamma\in\Gamma}]$ a ring of polynomials, and $Q \in \operatorname{Spec}^1(K[\Gamma])$. Then Q is a principal divisor of the form $K[\Gamma]q$, where q is an irreducible element of $K[\Gamma]$. Conversely, sets of the form given are members of* $\operatorname{Spec}^1(K[\Gamma])$.

Proof. We apply Proposition 2.2.4 to Q. If it is of form 1 then $Q = K_0[\Gamma]q \cap K[\Gamma] = K[\Gamma]q \cap K[\Gamma] = K[\Gamma]q$ as desired; it is clear that q must be irreducible in $K[\Gamma]$. If it is of form 2 then $Q \cap K \in \operatorname{Spec}^1(K) = \{K\}$, so $Q = K$; but then $Q = (Q \cap K)K[\Gamma] = K[\Gamma] \notin \operatorname{Spec}^1(K[\Gamma])$ (it certainly has height greater than 1 given $\Gamma = \emptyset$), a contradiction. The converse is trivial, so the result follows. ∎

And now, the main theorem for this section.

Theorem 5.1.2. *Let Γ be a set. Then there exists a Krull domain R with divisor class group* $\mathrm{Cl}(R) \cong \mathbb{Z}\{\Gamma\}$.

PROOF. We begin with a quick overview of the proof: First, we construct the ring. Second, we prove that it is a Krull domain. Third, we find an irredundant representation. Fourth, we describe the divisors and principal divisors. Fifth and finally, we determine the divisor class group.

Constructing R: Let K be a field. We successively define:

$$W = K[\{X_\gamma, Y_\gamma, Z_\gamma\}_{\gamma\in\Gamma}],$$

$$\begin{aligned} W_{\gamma_0} &= K[\{X_\gamma, Y_\gamma, Z_\gamma\}_{\gamma\in\Gamma,\gamma\neq\gamma_0}], \\ A_{\gamma_0} &= W_{\gamma_0}[X_{\gamma_0}, Y_{\gamma_0}, X_{\gamma_0}Z_{\gamma_0}] = W_{\gamma_0}[X_{\gamma_0}Z_{\gamma_0}][X_{\gamma_0}, Y_{\gamma_0}], \\ Q_{\gamma_0} &= A_{\gamma_0}X_{\gamma_0} + A_{\gamma_0}Y_{\gamma_0}, \\ v_{\gamma_0}(a) = k &\Leftrightarrow a \in Q_{\gamma_0}^k \setminus Q_{\gamma_0}^{k+1}, \text{ and} \\ R &= \bigcap_{\gamma\in\Gamma} (R_{v_\gamma} \cap W). \end{aligned}$$

Each of the rings $W, W_{\gamma_0}, A_{\gamma_0}$ is a K-algebra; Q_{γ_0} is an ideal of A_{γ_0}; v_{γ_0} is a discrete valuation on A_{γ_0} which can be extended to its field of fractions $(A_{\gamma_0})_0 = W_0$. Note that W is a Krull domain by Proposition 2.2.3, so we can express it as $W = \bigcap_{P\in\mathrm{Spec}^1(W)} W_P$. Then we have an alternative representation of R as

$$R = \left(\bigcap_{\gamma\in\Gamma} R_{v_\gamma}\right) \cap \left(\bigcap_{P\in\mathrm{Spec}^1(W)} W_P\right).$$

R is a Krull domain: Each R_{v_γ} and each W_P is a discrete valuation ring; therefore R satisfies the intersection property. To complete the proof that R is a Krull domain, we need to show that it also satisfies the finite character property.

Now, only finitely many indeterminates $X_\gamma, Y_\gamma, Z_\gamma$ occur in each non-zero element $r \in R$, so for almost all $\gamma \in \Gamma$, none of the indeterminates $X_\gamma, Y_\gamma, Z_\gamma$ occur in r. In these cases it follows that $v_\gamma(r) = 0$, so that r is invertible in R_{v_γ} for almost all $\gamma \in \Gamma$. Since W is a Krull domain it follows that r is invertible in almost all of the W_P as well. Therefore, R satisfies the finite character property as desired.

Irredundant representation: We note here as an aid to the reader that each ring $R_{v_\gamma} \cap W$ is isomorphic to $W_\gamma[X_\gamma, Y_\gamma, X_\gamma Z_\gamma, Y_\gamma Z_\gamma]$, so that $R \cong K[\{X_\gamma, Y_\gamma, X_\gamma Z_\gamma, Y_\gamma Z_\gamma\}_{\gamma\in\Gamma}]$; however, we choose to represent it as we do above in order to better display its characteristics as a Krull domain.

That being said, we claim that the representation

$$R = \left(\bigcap_{\gamma \in \Gamma} R_{v_\gamma} \right) \cap \left(\bigcap_{P \in \mathrm{Spec}^1(W)} W_P \right)$$

is irredundant. To prove this, it is sufficient to show that each term in the intersection restricts the result further, i.e. that there is an element that is not in that term, but is in every other. To do so, we consider three cases:

1. R_{v_γ}: Consider the term Z_γ; by the note above, it is easy to see that this element satisfies the required properties.
2. W_P, where $P = WX_\gamma$ for some $\gamma \in \Gamma$: Consider the term $\frac{Y_\gamma}{X_\gamma}$; this term has value $v_\gamma(\frac{Y_\gamma}{X_\gamma}) = 0$ for each $\gamma \in \Gamma$, so is in each R_{v_γ}. Note that X_γ is not contained in any prime ideal $P' \in \mathrm{Spec}^1(W)$ other than P ($X_\gamma \in P' \Rightarrow P = WX_\gamma \subseteq P' \Rightarrow P = P'$ since $\mathrm{ht}(P') = 1$); thus it is invertible in each of the $W_{P'}, P' \neq P$, and so our element is contained in each of those rings. It is clear that $\frac{Y_\gamma}{X_\gamma} \notin W_P$.
3. W_P, where $P \neq WX_\gamma$ for any $\gamma \in \Gamma$: Note that by Lemma 5.1.1 P is a principal divisor; let $P = Wf$ where $f \neq X_\gamma$ for any $\gamma \in \Gamma$. Let $\gamma_1, \ldots, \gamma_m$ be all the elements in Γ for which $v_\gamma(f) \neq 0$. Consider the element

   $$g = f^{-1} \prod_{i=1}^{m} X_{\gamma_i}^{\max(0, v_{\gamma_i}(f))}.$$

   It is clear that g is in $W_{P'}$ for all $P' \in \mathrm{Spec}^1(W), P' \neq P$ since f cannot lie in any of those P' (by the same argument as for X_γ above); it is also guaranteed to be in each R_{v_γ} by the use of the max function in the exponents. On the other hand, f cannot possibly divide $\prod_{i=1}^{m} X_{\gamma_i}^{\max(0, v_{\gamma_i}(f))}$ because each X_{γ_i} is prime and distinct from f (as assumed); thus, the multiplication by f^{-1} ensures that $g \notin W_P$ as desired.

Describing $\mathrm{Div}(R)$ and $\mathrm{Prin}(R)$: Using Proposition 2.2.5 with the irredundant representation of R given above, we have that $\mathrm{Spec}^1(R)$ consists of the all the ideals $\mathfrak{m}(R_{v_\gamma}) \cap R$, where $\gamma \in \Gamma$, and all the ideals $\mathfrak{m}(W_P) \cap R = PW_P \cap R = PW_P \cap (W \cap R) = (PW_P \cap W) \cap R = P \cap R$, where $P \in \mathrm{Spec}^1(W)$. It follows from Theorem 3.3.3 that $\mathrm{Div}(R) \cong \mathbb{Z}\{\{\mathfrak{m}(R_{v_\gamma}) \cap R : \gamma \in \Gamma\} \cup \{P \cap R : P \in \mathrm{Spec}^1(W)\}\} \cong \mathbb{Z}\{\Gamma \cup \mathrm{Spec}^1(W)\} \cong \mathbb{Z}\{\Gamma\} \oplus \mathbb{Z}\{\mathrm{Spec}^1(W)\}$.

We have seen that the isomorphism sends principal divisors $Rx \mapsto \mu_{Rx} = v(x)$, where $x \in R_0 = W_0, x \neq 0$; its representation in $\mathrm{Div}(R) \cong \mathbb{Z}\{\Gamma\} \oplus \mathbb{Z}\{\mathrm{Spec}^1(W)\}$ is thus

$$\mu_{Rx} = \left(\sum_{\gamma \in \Gamma} v_\gamma(x)\gamma \right) + \left(\sum_{P \in \mathrm{Spec}^1(W)} v_P(x)P \right),$$

so $\mathrm{Prin}(R) \cong \{\mu_{Rx} : x \in W_0 \setminus 0\} = \left\{ \left(\sum_{\gamma \in \Gamma} v_\gamma(x)\gamma \right) + \left(\sum_{P \in \mathrm{Spec}^1(W)} v_P(x)P \right) : x \in W_0 \setminus 0 \right\}$.

In summary, we have that

$$\mathrm{Div}(R) \cong \mathbb{Z}\{\Gamma\} \oplus \mathbb{Z}\{\mathrm{Spec}^1(W)\}, \text{ and}$$

$$\mathrm{Prin}(R) \cong \left\{ \left(\sum_{\gamma\in\Gamma} v_\gamma(x)\gamma \right) + \left(\sum_{P\in\mathrm{Spec}^1(W)} v_P(x)P \right) : x \in W_0 \diagdown 0 \right\}.$$

Determining $\mathrm{Cl}(R)$: By Lemma 5.1.1, we can express each $P \in \mathrm{Spec}^1(W)$ as Wf_P for some $f_P \in W \subseteq W_0 \subseteq R_0$. Then for any $P' \in \mathrm{Spec}^1(W), P' \neq P$, we have that $f_P \notin W_{P'}$, so it is invertible in each of those $W_{P'}$ and $v_{P'}(f_P) = 0$. It is also clear that $v_P(f_P) = 1$, so

$$\mu_{Rf_P} = \left(\sum_{\gamma\in\Gamma} v_\gamma(f_P)\gamma \right) + \left(\sum_{P\in\mathrm{Spec}^1(W)} v_P(f_P)P \right) = \left(\sum_{\gamma\in\Gamma} v_\gamma(f_P)\gamma \right) + P.$$

Now each $P \in \mathrm{Spec}^1(W)$ is a linear combination of μ_{Rf_P} and the elements of Γ, so the set of these elements together with Γ span $\mathrm{Div}(R) \cong \mathbb{Z}\{\Gamma\} \oplus \mathbb{Z}\{\mathrm{Spec}^1(W)\}$. If we let

$$H = \{\mu_{Rf_P} : P \in \mathrm{Spec}^1(W)\} = \left\{ \left(\sum_{\gamma\in\Gamma} v_\gamma(f_P)\gamma \right) + P : P \in \mathrm{Spec}^1(W) \right\}$$

then $\mathrm{Div}(R) \cong \mathbb{Z}\{\Gamma\} \oplus H$.

We now claim that $H = \mathrm{Prin}(R)$. It is clear from the definitions that $H \subseteq \mathrm{Prin}(R)$, so it suffices to show the opposite inclusion. Consider a principal divisor $\mu_{Rx} \in \mathrm{Prin}(R)$. Setting

$$y = x \left(\prod_{P\in\mathrm{Spec}^1(W)} f_P^{-v_P(x)} \right)$$

yields

$$v_{P'}(y) = v_{P'}(x) + \left(\sum_{P\in\mathrm{Spec}^1(W)} (-v_P(x) v_{P'} f_P) \right) = v_{P'}(x) - v'_P(x) = 0$$

for each $P' \in \mathrm{Spec}^1(W)$. Then y is an invertible element of W_P for every $P \in \mathrm{Spec}^1(W)$, whence y is an invertible element of $\bigcap_{P\in\mathrm{Spec}^1(W)} W_P = W$. Then $\mu_{Ry} = \mu_R = 0$, so

$$\begin{aligned}
\mu_{Rx} &= \mu_{R\left(y\left(\prod_{P\in\mathrm{Spec}^1(W)} f_P^{v_P(x)}\right)\right)} \\
&= \mu_{(Ry)\left(\prod_{P\in\mathrm{Spec}^1(W)} (Rf_P)^{v_P(x)}\right)} \\
&= \mu_{Ry} + \left(\sum_{P\in\mathrm{Spec}^1(W)} v_P(x)\mu_{Rf_P} \right) \\
&= \sum_{P\in\mathrm{Spec}^1(W)} v_P(x)\mu_{Rf_P} \\
&\in H
\end{aligned}$$

since each $\mu_{Rf_P} \in H$. Thus, $\mathrm{Prin}(R) \subseteq H$, whence $H = \mathrm{Prin}(R)$ as claimed.

Finally, we have $\mathrm{Cl}(R) = \mathrm{Div}(R)/\mathrm{Prin}(R) = \mathrm{Div}(R)/H \cong (\mathbb{Z}\{\Gamma\} \oplus H)/H \cong \mathbb{Z}\{\Gamma\}$ as desired, and the result is proven. $\square$

Thus we have established Step 1 of the construction.

5.2 Quotient Groups

In this section we demonstrate Step 2 of the construction. We begin with a few useful lemmas.

Lemma 5.2.1. *Let R be a normal domain, $r, s \in R$, $r \neq 0$, and $I = Rr^{-1} \cap Rs^{-1}$. Then $(rX - s)IR[X] = (rX - s)R_0[X] \cap R[X]$.*

PROOF. We show that each set is contained in the other.

$\subseteq$: Consider an element $(rX - s)yf$ in $(rX - s)IR[X]$, where $y \in I$ and $f \in R[X]$; it is sufficient to show that elements of this form are contained in $(rX-s)R_0[X] \cap R[X]$ since they span the ring. We have $y \in R_0 \Rightarrow yf \in R_0[X]$, so $(rX-s)yf \in (rX-s)R_0[X]$. Furthermore, $y \in I = Rr^{-1} \cap Rs^{-1} \Rightarrow (rX - s)y \in R[X]$, so $(rX - s)yf \in R[X]$. Therefore $(rX - s)yf \in (rX - s)R_0[X] \cap R[X]$, whence the entire set $(rX - s)IR[X] \subseteq (rX - s)R_0[X] \cap R[X]$ as desired.

$\supseteq$: Every element $f \in (rX - s)R_0[X] \cap R[X]$ is a polynomial of the form $(rX - s)t$, where $t \in R_0[X]$ is a polynomial whose coefficients are in R. We can represent this as

$$f = (rX - s)\sum_{i=0}^{m} t_i X^i,$$

so that f has leading term $rt_m X^{m+1}$. Then its coefficient rt_m must be an element of R.

Since substituting $\frac{1}{rt_m}X$ for X in f yields a polynomial $f(\frac{1}{rt_m}X)$ with leading term $\frac{1}{(rt_m)^{-m}}X^{m+1}$, the polynomial $g = (rt_m)^m f(\frac{1}{rt_m}X)$ is a monic polynomial with coefficients in R, i.e. $g \in R[X]$. Furthermore, $g(st_m) = (rt_m)^m f(\frac{st_m}{rt_m}) = (rt_m)^m f(\frac{s}{r}) = 0$ (since $rX - s$, and hence f, kills $\frac{s}{r}$), i.e. st_m is integral over R. But R is normal, so this implies that $st_m \in R$ as well.

Now knowing that $rt_m, st_m \in R$, we note that the polynomial

$$f' = f - (rX - s)t_m X^m = (rX - s)\sum_{i=0}^{m-1} t_i X^i$$

again has coefficients in R. Then by analagous proof to that above we can show that $rt_{m-1}, st_{m-1} \in R$ as well, and by induction $rt_i, st_i \in R$ for each i, $0 \leq i \leq m$. It follows that $t_i \in Rr^{-1} \cap Rs^{-1} = I$ for each i, $0 \leq i \leq m$, so that $f = (rX - s)t \in (rX - s)I[X] \subseteq (rX - s)IR[X]$ as desired. $\square$

Lemma 5.2.2. *Let R be a Krull domain and $\alpha \in \mathrm{Cl}(R[X])$. Then there exists a prime ideal $P \in \mathrm{Spec}^1(R[X])$ such that $\alpha = P + \mathrm{Prin}(R[X])$.*

PROOF. By Theorem 4.5.2 we have $\mathrm{Cl}(R) \cong \mathrm{Cl}(R[X])$, so we can find a divisor $J \in \mathrm{Div}(R)$ such that $J + \mathrm{Prin}(R) \mapsto \alpha$. By Theorem 3.4.2, we can write $J = Rx_0 \cap Rx_1$ ($x_0, x_1 \in R_0$) $= R\frac{a_0}{b_0} \cap R\frac{a_1}{b_1}$ ($a_0, b_0, a_1, b_1 \in R$) $= a_0a_1(R(a_1b_0)^{-1} \cap R(a_0b_1)^{-1})$, so J is of the form $t(Rr^{-1} \cap Rs^{-1})$ where $r, s, t \in R$. Take $I = Rr^{-1} \cap Rs^{-1} = (Rt^{-1})J$, where $Rt^{-1} \in \mathrm{Prin}(R)$, so that $I + \mathrm{Prin}(R) = J + \mathrm{Prin}(R) \mapsto \alpha$.

On the other hand, we know that $I + \mathrm{Prin}(R) \mapsto \mathrm{div}(IR[X]) + \mathrm{Prin}(R[X])$; by Corollary 4.3.5, $IR[X]$ is a divisor of $R[X]$, so this is just $IR[X] + \mathrm{Prin}(R[X])$. Then we have

$$\begin{aligned} \alpha &= IR[X] + \mathrm{Prin}(R[X]) \\ &= (rX - s)IR[X] + \mathrm{Prin}(R[X]) \text{ (again since } (rX - s)R[X] \in \mathrm{Prin}(R[X])) \\ &= (rX - s)R_0[X] \cap R[X] + \mathrm{Prin}(R[X]) \text{ (by Lemma 5.2.1).} \end{aligned}$$

Now, $rX - s$ is clearly irreducible in $R_0[X]$, so $(rX - s)R_0[X]$ is a prime of height 1 in $R_0[X]$. Then by Proposition 2.2.4 $(rX - s)R_0[X] \cap R[X] \in \mathrm{Spec}^1(R[X])$. Thus, letting $P = (rX - s)R_0[X] \cap R[X] \in \mathrm{Spec}^1(R[X])$ yields $P + \mathrm{Prin}(R[X]) = (rX - s)R_0[X] \cap R[X] + \mathrm{Prin}(R[X]) = \alpha$ as desired. $\square$

And now, the main theorem for this section.

Theorem 5.2.3. *Let R be a Krull domain with divisor class group G and H a subgroup of G. Then there exists a Krull domain R' with divisor class group $\mathrm{Cl}(R') \cong G/H$.*

PROOF. We begin with a quick overview of the proof: First, we construct the ring. Second, we prove that it is a Krull domain. Third, we determine the divisor class group.

Constructing R': We have that $\mathrm{Cl}(R[X]) \cong \mathrm{Cl}(R) \cong G$ from Theorem 4.5.2, so we can view the quotient map taking $G \to G/H$ as a map $\psi : \mathrm{Cl}(R[X]) \to G/H$ with $\mathrm{Ker}(\psi) = H$. By Lemma 5.2.2, we may choose for each $\xi \in \mathrm{Ker}(\psi)$ a prime ideal $P_\xi \in \mathrm{Spec}^1(R[X])$ such that $P + \mathrm{Prin}(R[X]) = \xi$. Let $\Lambda = \mathrm{Spec}^1(R[X]) \setminus \{P_\xi\}_{\xi \in \mathrm{Ker}(\psi)}$ and take R' to be the subintersection

$$R' = R[X](\Lambda) = \bigcap_{P \in \Lambda} (R[X])_P.$$

R' is a Krull domain: This follows from Proposition 2.2.2.

Determining $\mathrm{Cl}(R')$: By Theorem 4.3.7, the extension $i : R[X] \hookrightarrow R'$ induces a homomorphism $\mathrm{Cl}(i) : \mathrm{Cl}(R[X]) \to \mathrm{Cl}(R')$. By Theorem 4.4.2, this map is a surjection with kernel $(F + \mathrm{Prin}(R[X]))/\mathrm{Prin}(R[X])$, where F is the subgroup of $\mathrm{Div}(R)$ generated by the set $\mathrm{Spec}^1(R[X]) \setminus \Lambda = \mathrm{Spec}^1(R[X]) \setminus (\mathrm{Spec}^1(R[X]) \setminus \{P_\xi\}_{\xi \in \mathrm{Ker}(\psi)}) = \{P_\xi\}_{\xi \in \mathrm{Ker}(\psi)}$. Thus, we have that

$$\begin{aligned} \mathrm{Ker}(\mathrm{Cl}(i)) &= (F + \mathrm{Prin}(R[X]))/\mathrm{Prin}(R[X]) \\ &= (\mathbb{Z}\{P_\xi\}_{\xi \in \mathrm{Ker}(\psi)} + \mathrm{Prin}(R[X]))/\mathrm{Prin}(R[X]) \\ &= \mathbb{Z}\{P_\xi + \mathrm{Prin}(R[X])\}_{\xi \in \mathrm{Ker}(\psi)} \\ &= \mathbb{Z}\{\xi \in \mathrm{Ker}(\psi)\} \\ &= \mathrm{Ker}(\psi) \\ &= H, \end{aligned}$$

whence $\mathrm{Cl}(R') \cong \mathrm{Cl}(R[X])/\mathrm{Ker}(\mathrm{Cl}(i)) \cong G/H$ as desired. $\square$

Thus we have established Step 2 of the construction.

5.3 Constructions of Dedekind Domains

In this section we demonstrate Step 3 of the construction.

Theorem 5.3.1. *Let R be a Krull domain. Then there exists a Dedekind domain R' with ideal class group $\mathrm{Cl}(R') \cong \mathrm{Cl}(R)$.*

Proof. We begin with a quick overview of the proof: First, we construct the ring. Second, we prove that it is a Dedekind domain. Third, we determine the ideal class group.

Constructing R': Let $T = R[X_1, X_2, \ldots]$. From each prime ideal $Q \subseteq T$ of height greater than 1, choose a non-zero element r. It is only contained in finitely many height 1 prime ideals, so suppose $r \in P$ for $P = P_1, \ldots, P_m$, $r \notin P$ for $P \neq P_1, \ldots, P_m$. Then $Q \nsubseteq \bigcup_{i=1}^{m} P_i$ since otherwise $Q \subseteq P_i$ for some $i, 1 \leq i \leq m$, whence $\mathrm{ht}(P_i) \geq \mathrm{ht}(Q) > 1$, a contradiction. We choose $s \in Q \setminus \bigcup_{i=1}^{m} P_i$ and observe that there is some integer k for which $r, s \in R[X_1, \ldots, X_{k-1}]$; let $f_Q = rX_k - s$. Having defined f_Q for each Q, we let S be the multiplicative subset of T consisting of all finite products of the f_Q. Take $R' = T_S$, the localization of T at S.

R' is a Dedekind domain: T is a Krull domain by Proposition 2.2.3, whence T_S is a Krull domain by Proposition 2.2.1. It is now sufficient to show that T_S is dimension 1, since it would then follow from Theorem 2.3.3, that $R' = T_S$ is a Dedekind domain. But this is clear since

$$\begin{aligned}
\mathrm{Spec}(T_S) &= \mathrm{Spec}(T) \setminus \{P \in \mathrm{Spec}(T) : P \cap S \neq \emptyset\} \\
&= \mathrm{Spec}(T) \setminus \{P \in \mathrm{Spec}(T) : f_Q \in P \text{ for some } Q \in \mathrm{Spec}(T), \mathrm{ht}(Q) > 1\} \\
&\subseteq \mathrm{Spec}(T) \setminus \{Q \in \mathrm{Spec}(T) : \mathrm{ht}(Q) > 1\} \\
&= \mathrm{Spec}^1(T).
\end{aligned}$$

Since inclusions are also preserved under localization, these are still height 1 prime ideals, so T_S is dimension 1 as desired.

Determining $\mathrm{Cl}(R')$: First note that $\mathrm{Cl}(T) = \mathrm{Cl}(R[X_1, X_2, \ldots]) \cong \mathrm{Cl}(R)$ by Theorem 4.5.2. It is then sufficient to show that $f_Q T$ is a prime ideal of T for each $Q \in \mathrm{Spec}(T), \mathrm{ht}(Q) > 1$, so that by Theorem 4.5.1 $\mathrm{Cl}(R') = \mathrm{Cl}(T_S) \cong \mathrm{Cl}(T) \cong \mathrm{Cl}(R)$ as desired.

Fix Q and consider the construction of f_Q given above. Let $A = R[X_1, \ldots, X_{k-1}]$ and note that $A[X_k, \ldots] = T$. It is then sufficient to show that $f_Q A[X_k]$ is a prime ideal of $A[X_k]$, from which it follows that $f_Q A[X_k, \ldots]$ is a prime ideal of $A[X_k, \ldots]$, or equivalently, that $f_Q T$ is a prime ideal of T as desired.

Now, for all $P \in \mathrm{Spec}^1(T) \subseteq \mathrm{Spec}^1(A)$ we have

$$v_P(r) = 0 \text{ for } P \neq P_1, \dots, P_m \text{ while}$$

$$v_P(s) = 0 \text{ for } P = P_1, \dots, P_m;$$

also, $v_P(r), v_P(s) \geq 0$ for all $P \in \mathrm{Spec}^1(T) \subseteq \mathrm{Spec}^1(A)$ since $r, s \in T \supseteq A$. It follows that $v_P(rs) = \max(v_P(r), v_P(s))$ for all $P \in \mathrm{Spec}^1(T) \subseteq \mathrm{Spec}^1(A)$, i.e. $v(rs) = \max(v(r), v(s))$, or $\mu_{Rrs} = \max(\mu_{Rr}, \mu_{Rs})$. Then by Theorem 3.3.3, $Ars = Ar \cap As$, so that $A = Ar^{-1} \cap As^{-1}$. This allows us to apply Lemma 5.2.1, so that we have $f_Q A[X_k] = (rX_k - s)A_0[X_k] \cap A[X_k]$. But $rX_k - s$ is irreducible in $A_0[X_k]$, so $(rX_k - s)A_0[X_k] \cap A[X_k]$ is prime in $A[X_k]$ by Proposition 2.2.4, whence $f_Q A[X_k]$ is also prime as desired. □

Thus we have established Step 3 of the construction.

5.4 Abelian Groups as Ideal Class Groups

Finally, we are able to put our constructions together into our final result, an important application of Krull theory which uses all of the divisor theory we have developed thus far. We begin with an intermediate statement that is also of interest.

Theorem 5.4.1. *Let G be an abelian group. Then there exists a Krull domain R with divisor class group* $\mathrm{Cl}(R) \cong G$.

PROOF. Since G is abelian, we may represent it as the image of a homomorphism $\psi : \mathbb{Z}\{\Gamma\} \to G$. By Theorem 5.1.2, there exists a Krull domain R' with divisor class group $\mathrm{Cl}(R') \cong \mathbb{Z}\{\Gamma\}$. Since $\mathrm{Ker}(\psi)$ is a subgroup of $\mathbb{Z}\{\Gamma\} \cong \mathrm{Cl}(R')$, we can apply Theorem 5.2.3 to find a Krull domain R with divisor class group $\mathrm{Cl}(R) \cong \mathbb{Z}\{\Gamma\}/\mathrm{Ker}(\psi) \cong \mathrm{Im}(\psi) = G$, and the result is proven. □

Finally, we come to our main theorem:

Theorem 5.4.2 (Claborn's Theorem). *Let G be an abelian group. Then there exists a Dedekind domain R with ideal class group* $\mathrm{Cl}(R) \cong G$.

PROOF. First, we have from Theorem 5.4.1 that there exists a Krull domain R' with divisor class group $\mathrm{Cl}(R) \cong G$. Then, from Theorem 5.3.1, we have that there exists a Dedekind domain R with ideal class group $\mathrm{Cl}(R) \cong \mathrm{Cl}(R') \cong G$, and the result is proven. □

The importance of this theorem lies in its implications concerning factorization within Dedekind domains. Recall that a domain is a unique factorization domain if and only if its ideal class group is trivial, so that the size of the ideal class group can be used as an indicator of how far a domain is from being a unique factorization domain. Put differently, the size of the class group is a measure of non-uniqueness of factorization within a domain. Early results showed that the natural Dedekind domains to study — rings of algebraic integers in general, and number fields in particular — all have finite class groups. Because of this, the question of how large and how complicated these structures can actually be became an open problem for a long time. Claborn solved it in 1966 by proving the result above. By showing that ideal class groups can be infinite in size and have any abelian structure, the theorem implies that factorization becomes arbitrarily complex even in the relatively well-behaved family of Dedekind domains.

This concludes our study of constructions of rings with specified class groups, and with it the theory of divisors in Krull domains.

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